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The Imaging Properties of Electron Beams in Arbitrary Static Electromagnetic Fields

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THE IMAGING PROPERTIES OF ELECTRON BEAMS IN ARBITRARY STATIC ELECTROMAGNETIC FIELDS

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CONTENTS

	PAGE		PAGE
Notation and symbols	155	7. The zero-order chromatic aberration	170
1. Introduction	157	8. The second-order aberration	172
2. The metric	159	9. The paraxial chromatic aberration	178
3. The electric and magnetic potentials	160	10. Example: a helical beam in an electric field	180
4. The variational equation	164	Appendix. Change of aperture position	185
5. The paraxial ray equations	166	References	187
6. The orthogonality condition	168		

Electron-optical systems with curved axes—such as mass spectrographs and certain beta-ray spectrometers—have long been in practical use, but there has been available no complete theory of the aberrations of such systems. It is the object of the present paper to construct such a theory and to demonstrate, by an example, its application to practical problems.

An appropriate co-ordinate system is set up by means of a ray-axis together with its normal and binormal. The electric and magnetic fields are then investigated with the help of tensor calculus; the variational principle of electron optics is also put into tensor form. The integrand of the variational equation may be separated into a series of polynomials, one of which determines the paraxial imaging properties of the system and the rest of which determine the aberrations.

The condition is established for which, upon an appropriate transformation, either of the paraxial ray equations contains only one off-axis co-ordinate. Subsequent investigations are restricted to systems, which are termed 'orthogonal', for which this condition is satisfied. It is shown that, in a certain sense, no orthogonal electron-optical system can be wholly divergent.

The second-order aberration and the zero-order and paraxial chromatic aberrations are then investigated by the method of perturbation characteristic functions. All formulae are given in their relativistic forms but their non-relativistic forms are indicated; formulae are therefore given for the calculation of the zero-order and paraxial relativistic correction. It is indicated to what extent one forfeits control over the second-order aberration—and hence over the paraxial chromatic aberration also—by specifying that the paraxial behaviour of rays should be Gaussian.

As an example, the imaging properties of a helical beam moving in the field of a pair of coaxial cylindrical electrodes are calculated. There is also an appendix which gives formulae for the effect upon aberrations of a change in the aperture position.

Notation and symbols

(i) *German type* indicates tensor quantities.

(ii) *A suffix* attached to ϕ or Φ, as ϕ_i or Φ_x, indicates differentiation.

(iii) *A suffix following a comma*, as in $H_{y,x}$, denotes covariant differentiation.

† This work was initiated while the author was a member of the Electron Physics Section, National Bureau of Standards, Washington D.C.

(iv) *A prime*, as in κ', denotes differentiation with respect to z.

(v) *A superfix* '(r)', as in $m^{(1)}$, $m^{(2)}$, etc., denotes the order of the dependence of the quantity on the off-axis co-ordinates.

(vi) *A superfix* 'I', as in $*V^{(2)\mathrm{I}}_{ob}$, denotes the derivative of the original quantity with respect to the chromatic parameter ϵ.

(vii) *A superfix* 'R', as in $u^{(0)R}_{b}$, denotes the relativistic correction to the original quantity.

(viii) *The suffix* 'o', 'a', 'b', or 'c' indicates that a quantity is evaluated in the object, aperture, image or 'current' plane, respectively; a pair of such suffixes denote the limits of an integration.

(ix) *An asterisk*, as in D^*_1, converts a symbol appearing in chromatic-aberration calculations into its counterpart in relativistic-correction calculations.

(x) *Angular brackets* '$\langle\ \rangle$': we must take $\langle f\rangle = f$ or 1 according as our calculations are to be relativistically correct or non-relativistic, respectively.

In the following list of symbols, the section in which the symbol is introduced is indicated.

symbol	meaning
$\mathfrak{A}_x, \mathfrak{A}_y, \mathfrak{A}_z$	magnetic covariant vector potential (§ 2).
$\mathfrak{A}_1, \mathfrak{A}_2, \mathfrak{A}_3$	alternative notation for $\mathfrak{A}_x, \mathfrak{A}_y, \mathfrak{A}_z$ (§ 2).
A_x, A_y, A_z	values of $\mathfrak{A}_x, \mathfrak{A}_y, \mathfrak{A}_z$ on the ray-axis.
A_1, A_2	coefficients in expansion of $m^{(1)\mathrm{I}}$ in terms of x, y (§ 7).
B_1, B_2	coefficients in expansion of $m^{(1)\mathrm{I}}$ in terms of u, v (§ 7).
C_1, etc.	coefficients in the expansion of $*V^{(1)\mathrm{I}}$ in terms of u_o, v_o, u_a, v_a (§ 7).
D_1, D_2	coefficients of zero-order chromatic aberration (§ 7).
D^{I}, etc.	operators changing $f(u, v, u', v')$ to $u^{\mathrm{I}}\,\partial f/\partial u + \ldots$, etc. (§ 9).
E	takes the value 1 or 0 according as there is or is not an electric field (§ 5).
E_1, E_2	coefficients of the zero-order chromatic aberration in the image plane (§ 7).
f, g, h, k	functions determining the general paraxial ray (§ 6).
F_1, etc.	coefficients in the expansion of $m^{(3)}$ in terms of u, v, u', v' (§ 8).
g	see f.
g_{ij}	the metric tensor (§ 2).
G_1, etc.	functions from which F_1, etc., may be evaluated (§ 8).
h	see f and i.
$\mathfrak{H}_{ij}$	the magnetic field tensor (§ 3).
H_x, H_y, H_z	components of the magnetic field strength on the ray-axis (§ 3).
i, j, k, h	take the values 1, 2, 3 (§ 2).
j	see i.
k	see f and i.
K_1, etc.	coefficients in the expansion of $*V^{(3)}$ in terms of u_o, v_o, u_a, v_a (§ 8).
L_1, etc.	coefficients of the second-order aberration (§ 8).
m	the integrand of the variational equation (§§ 1, 4).
M	takes the values 1 or 0 according as there is or is not a magnetic field (§ 5).
M_1, etc.	coefficients of the second-order aberration in the image plane (§ 8).
n_u, n_v	ray variables (§ 7).

N_1, etc.	coefficients of the second-order aberration of the ray variables in the image plane (§ 8).
$\mathfrak{p}$	the scalar momentum of the electron beam (§ 4).
p	the value of $\mathfrak{p}$ on the ray-axis (§ 4).
P_1, etc.	coefficients in the expansion of $m^{(2)\mathrm{I}}$ in terms of u, v, u', v' (§ 9).
q, r, s, t	coefficients of the paraxial variational function (§ 6).
Q_u, Q_v	invariants formed from f, g, h, k (§ 6).
r	see q.
R_1, etc.	functions from which P_1, etc., may be evaluated (§ 9).
s	see q.
S_1, etc.	coefficients in the expansion of $*V_{ob}^{(2)\mathrm{I}}$ in terms of u_o, v_o, u_a, v_a (§ 9).
t	see q.
T_1, etc.	coefficients of the paraxial chromatic aberration in the image plane (§ 9).
u, v	co-ordinates measured with respect to the principal axes (§ 6).
U, V	coefficients of the orthogonal paraxial variational function (§ 6).
v	see u.
V	see U.
$*V$	characteristic function (§§ 7, 8, 9).
x, y	co-ordinates measured with respect to the normal and binormal to the ray-axis (§ 2).
x^1, x^2, x^3	alternative notation for x, y, z (§ 2).
y	see x.
z	co-ordinate measured along the ray-axis (§ 2).
$\alpha, \beta, \gamma, \theta$	modified forms of f, g, h, k (§ 6).
ϵ	small increase of beam energy (§ 4).
ϕ	electric potential (§ 3).
Φ	value of ϕ on the ray-axis (§ 3).
ψ	magnetic scalar potential (§ 3).
κ	curvature of ray-axis (§ 2).
τ	torsion of ray-axis (§ 2).
χ	angle between u-, v-axes and x-, y-axes (§ 6).

1. Introduction

Since the first papers of Busch (1926, 1927) on the focusing of electron beams in magnetic fields, the electron optics of fields of rotational symmetry has been thoroughly developed by Glaser (1933, 1935), Scherzer (Brüche & Scherzer 1934), Picht (1932) and many others. The importance of rotationally symmetrical fields, the applications of which are well known, is due principally to the ease and precision with which they may be realized, partly to the formation of Gaussian images and the absence of second-order aberrations, but in some measure also to their mathematical tractability.

There is another class of fields of great practical importance which comprises fields which may be termed 'mirror symmetrical', since in each case the electric and magnetic scalar potentials are symmetrical and antisymmetrical, respectively, about a plane. For such fields there is orthogonal—but not necessarily Gaussian—paraxial imaging and there are second-order aberrations; the theoretical treatment of such fields is considerably more complex than that of fields of the previous class.

There have been many investigations of particular fields of mirror symmetry such as those of Herzog (1934), Marschall (1944) and Hachenberg (1948) which deal with mass spectrometers; those of Svartholm & Siegbahn (1947) and Shull & Dennison (1947), which treat beta-ray spectrometers; and those of Coggeshall (1947) and Ploch & Walcher (1950), which investigate the important 'fringe effect'.

Gabor (1951) has recently been led to investigate another class of electron-optical system in which the electron beam moves in a helix under the influence of an electric field. Gabor's calculations, which are of particular interest since they consider the paraxial chromatic aberration, are discussed further in § 10.

The author was at one time engaged upon the design of an instrument employing a magnetic field which was of mirror symmetry but had no other simplifying characteristics. The theoretical treatment of this system has been extended to include electric fields and all assumptions of symmetry have been discarded. The present essay therefore treats the imaging properties of electron beams in arbitrary static electromagnetic fields, and it is hoped that the calculation of the properties of any electron-optical system with a curved axis may conveniently be based upon the theory to be set out.

Cotte (1938) was perhaps the first to treat at length the imaging of beams in arbitrary fields. The co-ordinate system is so chosen that the metric tensor is diagonal; the use of tensor calculus is thereby avoided, but the application of the theory to practical problems is complicated. A large part of the thesis is devoted to a thorough discussion of paraxial imaging properties, but the treatment of second-order aberrations is not so satisfactory. Wendt (1943), giving an independent treatment, has also dealt fully with the paraxial image formation but has not considered the second-order aberrations. The review article by Hutter (1948) follows closely the work of Wendt. MacColl (1941, 1943) has given an elegant but formal theory of the imaging properties of electron beams based upon the paraxial ray equations; again there is no consideration of the second-order aberrations.

Since any investigation of an optical system is incomplete without a discussion of its principal aberrations, we shall set out not only to obtain the paraxial properties, but also to give explicit formulae for the zero-order and paraxial chromatic aberrations and the second-order geometrical aberration; these will be found by the method of perturbation characteristic functions (Sturrock 1951).

The method is to set up the variational equation of electron optics in the form

$$\delta \int m \, \mathrm{d}z = 0, \tag{1·1}$$

where $m = m(u, v, u', v', z)$, z being the 'ray-axis' co-ordinate and u, v the 'off-axis' co-ordinates. If we expand m as a series of polynomials $m^{(r)}$ of order r in u, v, u', v', and if we allow for a small chromatic variation characterized by a parameter ϵ, we obtain an expansion of the form

$$m = m^{(2)} + m^{(3)} + \ldots + \epsilon m^{(1)\mathrm{I}} + \epsilon m^{(2)\mathrm{I}} + \ldots; \tag{1·2}$$

the term $m^{(0)}$ is of no importance and $m^{(1)}$ vanishes since the z-axis is a ray. The ray co-ordinates in the Gaussian image plane $z = z_b$ may similarly be expanded as series of homogeneous polynomials $u_b^{(r)}$, $v_b^{(r)}$ in the ray co-ordinates u_o, v_o, u_a, v_a in the object and aperture planes $z = z_o$ and $z = z_a$:

$$\left.\begin{aligned} u_b &= u_b^{(1)} + u_b^{(2)} + \ldots + \epsilon u_b^{(0)\mathrm{I}} + \epsilon u_b^{(1)\mathrm{I}} + \ldots, \\ v_b &= v_b^{(1)} + v_b^{(2)} + \ldots + \epsilon v_b^{(0)\mathrm{I}} + \epsilon v_b^{(1)\mathrm{I}} + \ldots. \end{aligned}\right\} \tag{1·3}$$

By a suitable choice of co-ordinates, the paraxial terms reduce to $u_b^{(1)} = f_b u_o$, $v_b^{(1)} = g_b v_o$. The terms $u_b^{(2)}$, $v_b^{(2)}$, $u_b^{(0)\mathrm{I}}$, $v_b^{(0)\mathrm{I}}$ and $u_b^{(1)\mathrm{I}}$, $v_b^{(1)\mathrm{I}}$ represent the second-order geometrical aberration, the zero-order chromatic aberration and the paraxial chromatic aberration, respectively, and it is the purpose of the paper quoted above to obtain their relationship to the functions $m^{(3)}$, $m^{(1)\mathrm{I}}$ and $m^{(2)\mathrm{I}}$.

The following analysis of the problem may therefore be divided into two parts: §§ 2 to 4 are devoted to setting up the variational equation in the required form and §§ 5 to 9 are given to an investigation of the individual terms of the expansion (1·2).

2. The metric

We shall adopt a curvilinear co-ordinate system (x, y, z) based on a curve called the 'ray-axis' which will, in the first instance, be supposed chosen arbitrarily (figure 1). Let O be a fixed point on the ray-axis and P an arbitrary point; then, if P is sufficiently close to the ray-axis, there is a unique plane through P which cuts the ray-axis normally in a point N. The z-co-ordinate of P may be measured by the arc-length, taken in a prescribed direction, between O and N. The remaining co-ordinates, x, y, may be determined by interpreting the normal and binormal to the curve at N as x- and y-axes, respectively, in the plane which contains them.

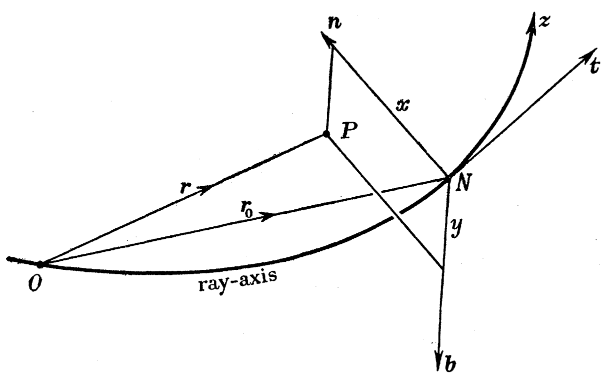

Figure 1.

If $\mathbf{r}_0(z)$ denotes the vector $\overrightarrow{ON}$ and $\mathbf{r}(x, y, z)$ the vector $\overrightarrow{OP}$,

$$\mathbf{r}(x, y, z) = \mathbf{r}_0(z) + x\mathbf{n}(z) + y\mathbf{b}(z), \tag{2·1}$$

where $\mathbf{t}(z)$, $\mathbf{n}(z)$ and $\mathbf{b}(z)$ will denote the unit vectors at N in the directions of the tangent, normal and binormal, respectively. We see from (2·1) that the change in the position vector due to increments in the co-ordinates is given by

$$\mathrm{d}\mathbf{r} = \mathbf{t}\,\mathrm{d}z + \mathbf{n}\,\mathrm{d}x + x\mathbf{n}'\,\mathrm{d}z + \mathbf{b}\,\mathrm{d}y + y\mathbf{b}'\,\mathrm{d}z, \tag{2·2}$$

where we denote by a prime differentiation with respect to z. On using the Serret-Frenet formulae (Weatherburn 1927, p. 15) and taking the scalar product of $\mathrm{d}\mathbf{r}$ with itself, we find that the metric of our co-ordinate system is determined by

$$\mathrm{d}s^2 = (1 - \kappa x)^2\,\mathrm{d}z^2 + (\mathrm{d}x - \tau y\,\mathrm{d}z)^2 + (\mathrm{d}y + \tau x\,\mathrm{d}z)^2. \tag{2·3}$$

It will be necessary, in this and the next sections, to make some use of the tensor calculus. We shall therefore introduce a dual notation, writing x, y, z alternatively as x^1, x^2, x^3, and writing the components of vectors such as $\mathfrak{A}_x$, $\mathfrak{A}_y$, $\mathfrak{A}_z$ as $\mathfrak{A}_1$, $\mathfrak{A}_2$, $\mathfrak{A}_3$ or $\mathfrak{A}^1$, $\mathfrak{A}^2$, $\mathfrak{A}^3$ according as the vector is covariant or contravariant, respectively. Numeral indices will imply the use of the summation convention.

The alternative notation makes it possible to rewrite (2·3) in the conventional form

$$ds^2 = \mathfrak{g}_{ij}\,dx^i\,dx^j, \tag{2·4}$$

where i and j and, later, k and h take the values 1, 2 and 3. The metric tensor $\mathfrak{g}_{ij}$ is given explicitly by

$$(\mathfrak{g}_{ij}) = \begin{pmatrix} 1 & 0 & -\tau y \\ 0 & 1 & \tau x \\ -\tau y & \tau x & (1-\kappa x)^2+\tau^2(x^2+y^2) \end{pmatrix}. \tag{2·5}$$

It is necessary for us to evaluate the Christoffel symbols of the second kind which are defined (Weatherburn 1938, p. 55) by

$$\begin{Bmatrix} k \\ ij \end{Bmatrix} = \tfrac{1}{2}\mathfrak{g}^{kh}\left\{\frac{\partial \mathfrak{g}_{jh}}{\partial x^i}+\frac{\partial \mathfrak{g}_{ih}}{\partial x^j}-\frac{\partial \mathfrak{g}_{ij}}{\partial x^h}\right\}. \tag{2·6}$$

However, it is necessary to note the values only of the functions and of their first derivatives on the ray-axis. If the superfix zero is used to indicate that a function is evaluated on the ray-axis, we find that

$$\left.\begin{aligned} &\begin{Bmatrix} y \\ xz \end{Bmatrix}^0 = \tau, \quad &&\begin{Bmatrix} z \\ xz \end{Bmatrix}^0 = -\kappa, \\ &\begin{Bmatrix} x \\ yz \end{Bmatrix}^0 = -\tau, \quad &&\begin{Bmatrix} x \\ zz \end{Bmatrix}^0 = \kappa, \end{aligned}\right\} \tag{2·7}$$

are the only symbols which are non-zero on the ray-axis. The non-zero values of the first derivatives are

$$\left.\begin{aligned} &\begin{Bmatrix} x \\ xz \end{Bmatrix}^0_y = -\kappa\tau, &&\begin{Bmatrix} y \\ xz \end{Bmatrix}^0_x = \kappa\tau, &&\begin{Bmatrix} y \\ xz \end{Bmatrix}^0_z = \tau', \\ &\begin{Bmatrix} z \\ xz \end{Bmatrix}^0_x = -\kappa^2, &&\begin{Bmatrix} z \\ xz \end{Bmatrix}^0_z = -\kappa', &&\begin{Bmatrix} x \\ yz \end{Bmatrix}^0_z = -\tau', \\ &\begin{Bmatrix} x \\ zz \end{Bmatrix}^0_x = -(\kappa^2+\tau^2), &&\begin{Bmatrix} x \\ zz \end{Bmatrix}^0_y = -\tau', &&\begin{Bmatrix} x \\ zz \end{Bmatrix}^0_z = \kappa', \\ &\begin{Bmatrix} y \\ zz \end{Bmatrix}^0_x = \tau', &&\begin{Bmatrix} y \\ zz \end{Bmatrix}^0_y = -\tau^2, \\ &\begin{Bmatrix} z \\ zz \end{Bmatrix}^0_x = -\kappa', &&\begin{Bmatrix} z \\ zz \end{Bmatrix}^0_y = \kappa\tau. \end{aligned}\right\} \tag{2·8}$$

3. The electric and magnetic potentials

As is pointed out in introductions to the tensor calculus (Weatherburn 1938, p. 58), the ordinary derivatives of the components of a vector or tensor are not, in general, components of a tensor; new tensors are formed by *covariant* differentiation. Therefore, when physical laws which involve spatial differentiation are expressed in tensor form, it is the covariant derivative—not the ordinary derivative—which appears. We shall therefore establish relations between the ordinary and covariant derivatives of the electric scalar and magnetic vector potentials, and set out the physical relations which exist between the components of the covariant derivatives.

Let us consider first the electric scalar potential $\phi(x,y,z)$; values of this function and of its derivatives will be written as $\Phi(z)$, $\Phi_x(z)$, etc., when they are evaluated on the ray-axis. The first derivatives, $\partial\phi/\partial x^i$, which will be written as ϕ_i or, alternatively, as ϕ_x, ϕ_y, ϕ_z, are the components of a covariant vector. If we denote covariant differentiation by a suffix following a comma, the second covariant derivative is defined (Weatherburn 1938, p. 59) by

$$\phi_{i,j} = \frac{\partial\phi_i}{\partial x^j} - \begin{Bmatrix} t \\ ij \end{Bmatrix}\phi_t. \tag{3·1}$$

Since the paired indices of the Christoffel symbols appearing in (2·7) and (2·8) always contain z, no distinction arises between ordinary and covariant differentiation unless differentiation with respect to z is involved. This is easy to understand, for the x- and y-axes by themselves form a Cartesian set. If we therefore omit the comma from derivatives involving x and y only, and if we again use a prime to denote ordinary differentiation with respect to z, we obtain from (3·1) the relations

$$\left.\begin{aligned} \Phi_{x,z} &= \kappa\Phi' + \Phi_x' - \tau\Phi_y, \\ \Phi_{y,z} &= \tau\Phi_x + \Phi_y', \\ \Phi_{z,z} &= \Phi'' - \kappa\Phi_x \end{aligned}\right\} \tag{3·2}$$

and, from the formula

$$\phi_{i,jk} = \frac{\partial^2\phi_i}{\partial x^j\,\partial x^k} - \begin{Bmatrix} t \\ ij \end{Bmatrix}_k \phi_t - \begin{Bmatrix} t \\ ij \end{Bmatrix}\frac{\partial\phi_t}{\partial x^k} - \begin{Bmatrix} t \\ ik \end{Bmatrix}\phi_{t,j} - \begin{Bmatrix} t \\ jk \end{Bmatrix}\phi_{i,t}, \tag{3·3}$$

the relations

$$\left.\begin{aligned} \Phi_{x,zz} &= \kappa'\Phi' + 2\kappa\Phi'' - (\kappa^2+\tau^2)\,\Phi_x + \Phi_x'' - \tau'\Phi_y - 2\tau\Phi_y' - \kappa\Phi_{xx}, \\ \Phi_{y,zz} &= \kappa\tau\Phi' + \tau'\Phi_x + 2\tau\Phi_x' - \tau^2\Phi_y + \Phi_y'' - \kappa\Phi_{xy}. \end{aligned}\right\} \tag{3·4}$$

Let us now consider the physical condition to be satisfied by the electric potential. In the absence of space-charge, this is the Laplace equation which, in tensor form, is

$$\mathfrak{g}^{ij}\phi_{i,j} = 0. \tag{3·5}$$

Remembering that covariant differentiations commute, that the covariant derivatives of the fundamental tensor vanish, and that this tensor reduces to the unit matrix on the ray-axis, we may obtain from (3·5) the relations

$$\left.\begin{aligned} \Phi_{yy} &= -\Phi_{xx} - \Phi_{z,z}, \\ \Phi_{xyy} &= -\Phi_{xxx} - \Phi_{x,zz}, \\ \Phi_{yyy} &= -\Phi_{xyy} - \Phi_{y,zz}. \end{aligned}\right\} \tag{3·6}$$

It is now possible, by means of (3·2), (3·4) and (3·6), to express all derivatives of ϕ of up to the third order, evaluated on the ray-axis, in terms of $\Phi(z)$, $\Phi_x(z)$, $\Phi_y(z)$, $\Phi_{xx}(z)$, $\Phi_{xy}(z)$, $\Phi_{xxx}(z)$ and $\Phi_{xxy}(z)$ and their ordinary z-derivatives. These functions may therefore be adopted as the data determining the electric field to the approximation here considered. The expansion for ϕ in terms of these coefficients is found to be

$$\begin{aligned}\phi(x,y,z) = {} & \Phi + x\Phi_x + y\Phi_y + \tfrac{1}{2}\{x^2\Phi_{xx} + 2xy\Phi_{xy} + y^2(-\Phi'' + \kappa\Phi_x - \Phi_{xx})\} \\ & + \tfrac{1}{6}\{x^3\Phi_{xxx} + 3x^2y\Phi_{xxy} + 3xy^2(-\kappa'\Phi' - 2\kappa\Phi'' \\ & \quad + (\kappa^2+\tau^2)\,\Phi_x - \Phi_x'' + \tau'\Phi_y + 2\tau\Phi_y' + \kappa\Phi_{xx} - \Phi_{xxx}) \\ & \quad + y^3(-\kappa\tau\Phi' - \tau'\Phi_x - 2\tau\Phi_x' + \tau^2\Phi_y - \Phi_y'' + \kappa\Phi_{xy} - \Phi_{xxy})\} \\ & + \dots \end{aligned} \tag{3·7}$$

This expansion may serve to relate the coefficients which we have adopted to the electrodes defining the electric field.

Let us now consider the magnetic field. Since it is the vector potential and not the scalar potential which appears in the variational equation of electron optics, we must introduce the components of the magnetic vector (since $\mu = 1$, **H** and **B** are identical) by means of the skew-symmetric tensor which is derived from the vector potential $\mathfrak{A}_i$ according to

$$\mathfrak{H}_{ij} = \mathfrak{A}_{j,i} - \mathfrak{A}_{i,j}. \tag{3·8}$$

The covariant derivative is related to the ordinary derivative by

$$\mathfrak{A}_{i,j} = \frac{\partial \mathfrak{A}_i}{\partial x^j} - \begin{Bmatrix} t \\ ij \end{Bmatrix} \mathfrak{A}_t. \tag{3·9}$$

It follows from (3·9) that (3·8) may also be written as

$$\mathfrak{H}_{ij} = \frac{\partial \mathfrak{A}_j}{\partial x^i} - \frac{\partial \mathfrak{A}_i}{\partial x^j}. \tag{3·10}$$

The covariant and ordinary first derivatives of the skew-symmetric tensor are related (Weatherburn 1938, p. 62) by

$$\mathfrak{H}_{ij,k} = \frac{\partial \mathfrak{H}_{ij}}{\partial x^k} - \begin{Bmatrix} t \\ ik \end{Bmatrix} \mathfrak{H}_{tj} - \begin{Bmatrix} t \\ kj \end{Bmatrix} \mathfrak{H}_{it}. \tag{3·11}$$

On evaluating this formula on the ray-axis by means of the formulae (2·7), one finds, with the notation

$$H_x(z) = \mathfrak{H}_{yz}(0,0,z), \quad H_y(z) = \mathfrak{H}_{zx}(0,0,z), \quad H_z(z) = \mathfrak{H}_{xy}(0,0,z), \tag{3·12}$$

that

$$\left.\begin{aligned} H_{x,x} &= \frac{\partial H_x}{\partial x} + \kappa H_x, & H_{x,y} &= \frac{\partial H_x}{\partial y} - \tau H_z, & H_{x,z} &= \frac{\partial H_x}{\partial z} + \kappa H_z - \tau H_y, \\ H_{y,x} &= \frac{\partial H_y}{\partial x} + \tau H_z + \kappa H_y, & H_{y,y} &= \frac{\partial H_y}{\partial y}, & H_{y,z} &= \frac{\partial H_y}{\partial z} + \tau H_x, \\ H_{z,x} &= \frac{\partial H_z}{\partial x}, & H_{z,y} &= \frac{\partial H_z}{\partial y}, & H_{z,z} &= \frac{\partial H_z}{\partial z} - \kappa H_x. \end{aligned}\right\} \tag{3·13}$$

Since the metric tensor reduces to the unit matrix on the ray-axis, H_x, H_y, H_z are the components of magnetic field strength on the ray-axis.

The second derivatives are related by the formula

$$\mathfrak{H}_{ij,kh} = \frac{\partial^2 \mathfrak{H}_{ij}}{\partial x^k \partial x^h} - \begin{Bmatrix} t \\ ik \end{Bmatrix}_h \mathfrak{H}_{tj} - \begin{Bmatrix} t \\ ik \end{Bmatrix} \frac{\partial \mathfrak{H}_{tj}}{\partial x^h} - \begin{Bmatrix} t \\ jk \end{Bmatrix}_h \mathfrak{H}_{it} - \begin{Bmatrix} t \\ jk \end{Bmatrix} \frac{\partial \mathfrak{H}_{it}}{\partial x^h} - \begin{Bmatrix} t \\ ih \end{Bmatrix} \mathfrak{H}_{tj,k} - \begin{Bmatrix} t \\ jh \end{Bmatrix} \mathfrak{H}_{it,k} - \begin{Bmatrix} t \\ kh \end{Bmatrix} \mathfrak{H}_{ij,t}. \tag{3·14}$$

From (3·14), (2·7) and (2·8) we may obtain relations which may, with the help of (3·13), be written as

$$\left.\begin{aligned} \frac{\partial^2 H_x}{\partial x^2} &= H_{x,xx}-2\kappa H_{x,x}, & \frac{\partial^2 H_x}{\partial x\,\partial y} &= H_{x,xy}+\tau H_{z,x}-\kappa H_{x,y}, \\ \frac{\partial^2 H_x}{\partial y^2} &= H_{x,yy}+2\tau H_{z,y}, & \frac{\partial^2 H_y}{\partial x^2} &= H_{y,xx}-2\tau H_{z,x}-2\kappa H_{y,x}, \\ \frac{\partial^2 H_y}{\partial x\,\partial y} &= H_{y,xy}-\tau H_{z,y}-\kappa H_{y,y}, & \frac{\partial^2 H_y}{\partial y^2} &= H_{y,yy}. \end{aligned}\right\} \tag{3·15}$$

We need also the following two relations:

$$\left.\begin{aligned} H_{x,zz} &= -(\kappa^2+\tau^2)\,H_x+H_x''-\tau' H_y-2\tau H_y'+\kappa' H_z+2\kappa H_z'-\kappa H_{x,x}, \\ H_{y,zz} &= \tau' H_x+2\tau H_x'-\tau^2 H_y+H_y''+\kappa\tau H_z-\kappa H_{y,x}. \end{aligned}\right\} \tag{3·16}$$

Let us now consider the relevant field equations. The homogeneous field equations, which follow from the skew-symmetry of $\mathfrak{H}_{ij}$, are

$$\mathfrak{H}_{ij,k}+\mathfrak{H}_{jk,i}+\mathfrak{H}_{ki,j} = 0. \tag{3·17}$$

The equations (3·17) reduce, on the ray-axis, to

$$H_{x,x}+H_{y,y}+H_{z,z} = 0. \tag{3·18}$$

We may also confirm, by taking covariant derivatives of the tensor equation (3·17), that

$$\left.\begin{aligned} H_{x,xx}+H_{y,yx}+H_{z,zx} &= 0, \\ H_{x,xy}+H_{y,yy}+H_{z,zy} &= 0, \\ H_{x,xz}+H_{y,yz}+H_{z,zz} &= 0. \end{aligned}\right\} \tag{3·19}$$

In the absence of space-current, the inhomogeneous field equations may be written in the tensor form

$$\mathfrak{g}^{jk}\mathfrak{H}_{ij,k} = 0; \tag{3·20}$$

since the covariant derivatives of the metric tensor vanish, we have also

$$\mathfrak{g}^{jk}\mathfrak{H}_{ij,kh} = 0. \tag{3·21}$$

On evaluating (3·20) and (3·21) on the ray-axis, we obtain a number of equations of the forms $H_{y,z} = H_{z,y}$ and $H_{y,zx} = H_{z,yx}$ which may be summarized by the statement '*if two derivatives* (such as $H_{x,yz}$ and $H_{z,xy}$) *may be interchanged by permutation of their suffixes, they are equal in value.*'

It is now possible, by means of the above rule and relations, to select a suitable set of functions as the data determining the magnetic field to the approximation required. We adopt the functions $H_x(z)$, $H_y(z)$, $H_z(z)$, $H_{y,x}(z)$, $H_{y,y}(z)$, $H_{y,xx}(z)$ and $H_{y,xy}(z)$. It is found that the magnetic scalar potential is related to this set of coefficients by the expansion

$$\begin{aligned} \psi(x,y,z)-\psi(0,0,z) = &-\{xH_x+yH_y\}+\tfrac{1}{2}\{x^2(-\kappa H_x-H_z'-H_{y,y})+2xyH_{y,x}+y^2H_{y,y}\} \\ &-\tfrac{1}{6}\{x^3((\kappa^2+\tau^2)\,H_x-H_x''+\tau' H_y+2\tau H_y'-\kappa' H_z-2\kappa' H_z'+\kappa H_{x,x}-H_{y,xy}) \\ &\quad+3x^2yH_{y,xx}+3xy^2H_{y,xy} \\ &\quad+y^3(-\tau' H_x-2\tau H_x'+\tau^2 H_y-H_y''-\kappa\tau H_z+\kappa H_{y,x}-H_{y,xx})\} \\ &-\dots \end{aligned} \tag{3·22}$$

together with the equation

$$\frac{\partial}{\partial z}\psi(0,0,z) = -H_z(z). \tag{3·23}$$

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

4. The variational equation

It is well known (Glaser 1933) that the paths of monokinetic electrons in a static electromagnetic field may be derived from a variational principle formally identical with Fermat's principle of light optics. The equation may be written in tensor form as

$$\delta \int \{\mathfrak{p}\,(\mathfrak{g}_{ij}\,\dot{x}^i \dot{x}^j)^{\frac{1}{2}} - \dot{x}^i \mathfrak{A}_i\}\,\mathrm{d}t = 0, \tag{4·1}$$

where $\mathfrak{p}$ is the scalar kinetic momentum of the beam, measured in the same units as the vector potential, t is an arbitrary parameter of integration, a dot denotes differentiation with respect to t, and δ refers to variations of the integral due to arbitrary small variations of the path of integration between fixed end-points.

If we adopt the following units:

unit of electric potential = 511200 volts,

unit of magnetic potential = 1704 gauss cm,

the relation between the kinetic momentum, measured in magnetic units, and the local electric potential becomes

$$\mathfrak{p} = \sqrt{(2\phi + \phi^2)}, \tag{4·2}$$

the zero-point of ϕ being so chosen that electrons are at rest at zero potential. The expression (4·2) is 'relativistically correct'; the 'non-relativistic approximation' is obtained by replacing (4·2) by

$$\mathfrak{p} = \sqrt{(2\phi)}. \tag{4·2$'$}$$

Calculations will be based on (4·2) but the non-relativistic forms of important formulae will be indicated. If the system is purely electric and relativistic effects are of no interest, or if it is purely magnetic and the beam energy is measured by p rather than Φ, subsequent calculations will remain valid for any choice of units.

If we adopt the co-ordinate system of § 2 and adopt z as the parameter of integration, (4·1) may be written as

$$\delta \int m\,\mathrm{d}z = 0, \tag{4·3}$$

where

$$m = \mathfrak{p}\{(1-\kappa x)^2 + (x' - \tau y)^2 + (y' + \tau x)^2\}^{\frac{1}{2}} - (x'\mathfrak{A}_x + y'\mathfrak{A}_y + \mathfrak{A}_z). \tag{4·4}$$

The variational function $m(x', y', x, y, z)$ may be expanded as

$$m = m^{(0)} + m^{(1)} + m^{(2)} + \dots, \tag{4·5}$$

wherein $m^{(r)}$ denotes a homogeneous polynomial of the rth degree in x', y', x and y, the coefficients being functions of z. On expanding ϕ, $\mathfrak{A}_x$, $\mathfrak{A}_y$ and $\mathfrak{A}_z$ as Taylor series in x and y, we obtain

$$m^{(1)} = -p\kappa x + p^{-1}(1+\Phi)\,(x\Phi_x + y\Phi_y) - \left(x'A_x + y'A_y + x\frac{\partial A_z}{\partial x} + y\frac{\partial A_z}{\partial y}\right), \tag{4·6}$$

where A_x, A_y, A_z are the values of $\mathfrak{A}_x$, $\mathfrak{A}_y$, $\mathfrak{A}_z$ on the ray-axis, and similar, but longer, expressions for $m^{(2)}$ and $m^{(3)}$. The function $p(z)$ is derived from

$$p = \sqrt{(2\Phi + \Phi^2)} \tag{4·7}$$

for relativistically correct calculations and from

$$p = \surd(2\Phi) \tag{4·7'}$$

for non-relativistic calculations. The function $m^{(0)}$ may be ignored.

On integrating by parts terms involving x' and y', we may replace coefficients of the vector potential by coefficients of the magnetic vector. If we then make use of the relations (3·2), (3·4), (3·6), (3·13), (3·15), (3·16), (3·18) and (3·19), we may express all field coefficients in terms of the sets proposed in § 3 and so obtain the formulae

$$m^{(1)} = \{-p\kappa + p^{-1}(1+\Phi)\,\Phi_x + H_y\}\,x + \{p^{-1}(1+\Phi)\,\Phi_y - H_x\}\,y, \tag{4·8}$$

$$\left.\begin{aligned} m^{(2)} = {} & \tfrac{1}{2}p(x'^2+y'^2) + \tfrac{1}{2}(x'y-xy')\,(H_z - 2p\tau) \\ & + \tfrac{1}{2}x^2\{p\tau^2 - 2p^{-1}\kappa(1+\Phi)\,\Phi_x - p^{-3}\Phi_x^2 + p^{-1}(1+\Phi)\,\Phi_{xx} - \kappa H_y - \tau H_z + H_{y,\,x}\} \\ & + \tfrac{1}{2}xy\{-2p^{-1}\kappa(1+\Phi)\,\Phi_y - 2p^{-3}\Phi_x\Phi_y + 2p^{-1}(1+\Phi)\,\Phi_{xy} + H_z' + 2H_{y,\,y}\} \\ & + \tfrac{1}{2}y^2\{p\tau^2 - p^{-1}(1+\Phi)\,\Phi'' + p^{-1}\kappa(1+\Phi)\,\Phi_x - p^{-3}\Phi_y^2 - p^{-1}(1+\Phi)\,\Phi_{xx} - \tau H_z - H_{y,\,x}\}, \end{aligned}\right\} \tag{4·9}$$

and

$$\left.\begin{aligned} m^{(3)} = {} & \tfrac{1}{2}x(x'^2+y'^2)\,\{p\kappa + p^{-1}(1+\Phi)\,\Phi_x\} + \tfrac{1}{2}y(x'^2+y'^2)\,p^{-1}(1+\Phi)\,\Phi_y \\ & + \tfrac{1}{3}x(x'y-xy')\,\{-3p\kappa\tau - 3p^{-1}\tau(1+\Phi)\,\Phi_x + H_x' - \tau H_y + \kappa H_z\} \\ & + \tfrac{1}{3}y(x'y-xy')\,\{-3p^{-1}\tau(1+\Phi)\,\Phi_y + \tau H_x + H_y'\} \\ & + \tfrac{1}{6}x^3\{3p\kappa\tau^2 + 3p^{-1}\tau^2(1+\Phi)\,\Phi_x + 3p^{-3}\kappa\Phi_x^2 + 3p^{-5}(1+\Phi)\,\Phi_x^3 - 3p^{-1}\kappa(1+\Phi)\,\Phi_{xx} \\ & \quad - 3p^{-3}\Phi_x\Phi_{xx} + p^{-1}(1+\Phi)\,\Phi_{xxx} - 2\tau H_x' + 2\tau^2 H_y - 2\kappa\tau H_z - 2\kappa H_{y,\,x} + H_{y,\,xx}\} \\ & + \tfrac{1}{6}x^2y\{3p^{-1}\tau^2(1+\Phi)\,\Phi_y + 6p^{-3}\kappa\Phi_x\Phi_y + 9p^{-5}(1+\Phi)\,\Phi_x^2\Phi_y - 6p^{-1}\kappa(1+\Phi)\,\Phi_{xy} \\ & \quad - 6p^{-3}\Phi_x\Phi_{xy} - 3p^{-3}\Phi_y\Phi_{xx} + 3p^{-1}(1+\Phi)\,\Phi_{xxy} - 3\tau^2 H_x + H_x'' - \tau' H_y \\ & \quad - 4\tau H_y' + \kappa' H_z + \kappa H_z' - 3\kappa H_{y,\,y} + 3H_{y,\,xy}\} \\ & + \tfrac{1}{6}xy^2\{3p\kappa\tau^2 - 3p^{-1}\kappa'(1+\Phi)\,\Phi' - 3p^{-1}\kappa(1+\Phi)\,\Phi'' + 6p^{-1}\tau^2(1+\Phi)\,\Phi_x \\ & \quad + 3p^{-3}\Phi''\Phi_x - 3p^{-1}(1+\Phi)\,\Phi_x'' + 3p^{-1}\tau'(1+\Phi)\,\Phi_y + 6p^{-1}\tau(1+\Phi)\,\Phi_y' \\ & \quad - 3p^{-3}\kappa\Phi_x^2 + 3p^{-3}\kappa\Phi_y^2 + 9p^{-5}(1+\Phi)\,\Phi_x\Phi_y^2 + 6p^{-1}\kappa(1+\Phi)\,\Phi_{xx} \\ & \quad + 3p^{-3}\Phi_x\Phi_{xx} - 6p^{-3}\Phi_y\Phi_{xy} - 3p^{-1}(1+\Phi)\,\Phi_{xxx} - \tau' H_x - 4\tau H_x' \\ & \quad + 3\tau^2 H_y - H_y'' - 3\kappa\tau H_z + 3\kappa H_{y,\,x} - 3H_{y,\,xx}\} \\ & + \tfrac{1}{6}y^3\{-p^{-1}\kappa\tau(1+\Phi)\,\Phi' - p^{-1}\tau'(1+\Phi)\,\Phi_x - 2p^{-1}\tau(1+\Phi)\,\Phi_x' + 4p^{-1}\tau^2(1+\Phi)\,\Phi_y \\ & \quad + 3p^{-3}\Phi''\Phi_y - p^{-1}(1+\Phi)\,\Phi_y'' + 3p^{-5}(1+\Phi)\,\Phi_y^3 - 3p^{-3}\kappa\Phi_x\Phi_y \\ & \quad + p^{-1}\kappa(1+\Phi)\,\Phi_{xy} + 3p^{-3}\Phi_y\Phi_{xx} - p^{-1}(1+\Phi)\,\Phi_{xxy} - 2\tau^2 H_x - 2\tau H_y' - H_{y,\,xy}\}. \end{aligned}\right\} \tag{4·10}$$

We may also deduce from these formulae the functions which lead to the zero-order and paraxial chromatic aberration. Suppose that the beam energy is increased by an amount ϵ, measured as an equivalent increase in electric potential. If we regard ϵ as a perturbation parameter and adopt the notation of the recent paper on perturbation characteristic functions (Sturrock 1951), we find that

$$m^{(1)\mathrm{I}} = \{-p^{-1}\kappa(1+\Phi) - p^{-3}\Phi_x\}\,x - p^{-3}\Phi_y\,y \tag{4·11}$$

and

$$m^{(2)\mathrm{I}} = \tfrac{1}{2}(x'^2+y'^2)\,p^{-1}(1+\Phi)-(x'y-xy')\,p^{-1}\tau(1+\Phi)$$
$$+\tfrac{1}{2}x^2\{p^{-1}\tau^2(1+\Phi)+2p^{-3}\kappa\Phi_x+3p^{-5}(1+\Phi)\,\Phi_x^2-p^{-3}\Phi_{xx}\}$$
$$+xy\{p^{-3}\kappa\Phi_y+3p^{-5}(1+\Phi)\,\Phi_x\Phi_y-p^{-3}\Phi_{xy}\}$$
$$+\tfrac{1}{2}y^2\{p^{-1}\tau^2(1+\Phi)+p^{-3}\Phi''-p^{-3}\kappa\Phi_x+3p^{-5}(1+\Phi)\,\Phi_y^2+p^{-3}\Phi_{xx}\}. \qquad (4\cdot12)$$

Our calculations have so far been relativistically correct, being based on the formula (4·2) rather than (4·2′), but it is not difficult to see that the above formulae may be made non-relativistic by replacing the factor $(1+\Phi)$ by unity. It will be convenient to introduce a notation by means of which the non-relativistic form of a relativistic formula can easily be derived. We shall therefore introduce the 'angular' brackets '$\langle\ \rangle$' and understand that *the non-relativistic form of any formula may be obtained by replacing all terms contained in angular brackets by unity*. For this reason, the factor $(1+\Phi)$ will always be found written as $\langle 1+\Phi\rangle$. With this notation, the formulae (4·7) and (4·7′) may be combined in the formula

$$p^2 = 2\Phi\langle 1+\tfrac{1}{2}\Phi\rangle. \qquad (4\cdot13)$$

In dealing with systems—such as mass spectrographs—for which the field potentials are no higher than 2000 V or so, the non-relativistic approximation is found satisfactory; for field potentials above about 100000 V, the relativistic treatment is necessary. Between these limits, however, one can satisfactorily treat the 'relativistic correction' by perturbation methods.

Let us consider the electron-optical system defined by (4·1) in which $\mathfrak{p}$ is now replaced by $\mathfrak{p}^*$ defined by

$$\mathfrak{p}^* = \surd(2\phi+\sigma\phi^2), \qquad (4\cdot14)$$

where σ is a parameter. The relativistic formula (4·2) is given by $\sigma=1$, the non-relativistic formula (4·2′) by $\sigma=0$; the first-order relativistic correction may therefore be obtained by regarding σ as a perturbation parameter and evaluating the first-order perturbation for the particular value $\sigma=1$. The appropriate variational functions are found to be

$$m^{(1)R} = \tfrac{1}{8}\{-p^3\kappa+3p\Phi_x\}\,x+\tfrac{3}{8}p\Phi_y y \qquad (4\cdot15)$$

and

$$m^{(2)R} = \tfrac{1}{16}p^3(x'^2+y'^2)-\tfrac{1}{8}p^3\tau(x'y-xy')$$
$$+\tfrac{1}{16}x^2\{p^3\tau^2-6p\kappa\Phi_x+3p^{-1}\Phi_x^2+3p\Phi_{xx}\}$$
$$+\tfrac{1}{8}xy\{-3p\kappa\Phi_y+3p^{-1}\Phi_x\Phi_y+3p\Phi_{xy}\}$$
$$+\tfrac{1}{16}y^2\{p^3\tau^2-3p\Phi''+3p\kappa\Phi_x+3p^{-1}\Phi_y^2-3p\Phi_{xx}\}, \qquad (4\cdot16)$$

wherein the superfix 'R' has been adopted to denote the first-order relativistic correction, and p is to be derived from (4·7′).

5. The paraxial ray equations

In the last section we established a number of variational functions which will determine certain of the imaging properties of electron beams in arbitrary fields. In the present section we shall concern ourselves with the functions $m^{(1)}$ and $m^{(2)}$ which are given by (4·8) and (4·9) respectively.

Let us consider the condition that the ray-axis should be a ray traversing the field. This is clearly the condition that the variational equation (4·3) should have $x = y = 0$ as a solution which is satisfied if and only if $m^{(1)}$, expressed as a linear combination of x and y, vanishes identically. Hence, from (4·8),

$$\left.\begin{aligned} -p\kappa + p^{-1}\langle 1+\Phi\rangle \Phi_x + H_y &= 0,\\ p^{-1}\langle 1+\Phi\rangle \Phi_y - H_x &= 0.\end{aligned}\right\} \tag{5·1}$$

The relations (5·1) may be regarded as formulae for Φ_x and Φ_y or for H_x and H_y; our choice will obviously depend upon whether the field is purely electric, purely magnetic, or combined electric and magnetic, and it is therefore necessary to introduce a notation which will distinguish between these cases. The introduction of a suitable notation will also make it easier to draw from general formulae the particular case, which often arises in practice, that the field is either electric or magnetic.

Let us introduce the two parameters E and M defined as follows:†

$$\left.\begin{aligned} \mathsf{E} &= 0 \text{ if no electric field is present,}\\ \mathsf{E} &= 1 \text{ if an electric field is present;}\\ \mathsf{M} &= 0 \text{ if no magnetic field is present,}\\ \mathsf{M} &= 1 \text{ if a magnetic field is present.}\end{aligned}\right\} \tag{5·2}$$

We may note in particular that

$$\left.\begin{aligned} 1-\mathsf{M} &= 0 \text{ if the field is not purely electric,}\\ 1-\mathsf{M} &= 1 \text{ if the field is purely electric,}\end{aligned}\right\} \tag{5·3}$$

and a similar rule for $1-\mathsf{E}$. Henceforth it will be understood that in any formula which contains E and M we should, if the field is purely electric, set $\mathsf{E} = 1$, $\mathsf{M} = 0$; if the field is purely magnetic, set $\mathsf{E} = 0$, $\mathsf{M} = 1$; and, if the field is combined electric and magnetic, set $\mathsf{E} = 1$, $\mathsf{M} = 1$.

It is now proposed that we treat (5·1) as formulae for Φ_x and Φ_y if the field is purely electric, but as formulae for H_x and H_y otherwise. Thus

$$\left.\begin{aligned} &\left.\begin{aligned} \Phi_x &= p^2\kappa\langle 1+\Phi\rangle^{-1},\\ \Phi_y &= 0\end{aligned}\right\} \text{ if } \mathsf{M} = 0,\\ &\left.\begin{aligned} H_x &= \mathsf{E}p^{-1}\langle 1+\Phi\rangle \Phi_y,\\ H_y &= p\kappa - \mathsf{E}p^{-1}\langle 1+\Phi\rangle \Phi_x\end{aligned}\right\} \text{ if } \mathsf{M} = 1.\end{aligned}\right\} \tag{5·4}$$

The formulae (5·4) will be taken into account in all subsequent calculations.

Since $m^{(1)}$ vanishes, the integrand m of (4·3) tends to $m^{(2)}$ as x', y', x and y decrease. If we exclude from the present paper the discussion of electron mirrors, so that x' and y' become small as x and y become small, the 'paraxial' properties of electron beams may be derived

† This notation was suggested by Mr D. J. Behrens of A.E.R.E.

from $m^{(2)}$. The paraxial ray equations are therefore the Lagrangian equations derivable from this function which are found to be

$$\left.\begin{aligned}
&\frac{d}{dz}\left(p\frac{dx}{dz}\right)+(\mathsf{M}H_z-2p\tau)\frac{dy}{dz}\\
&\qquad+\{p\kappa^2-p\tau^2+(1-\mathsf{M})\,p\kappa^2(1+\langle 1+\Phi\rangle^{-2})-\mathsf{E}p^{-1}\langle 1+\Phi\rangle\,\Phi_{xx}\\
&\qquad\quad+\mathsf{M}(\tau H_z-H_{y,x})+\mathsf{EM}(p^{-1}\kappa\langle 1+\Phi\rangle\,\Phi_x+p^{-3}\Phi_x^2)\}\,x\\
&\qquad+\{-p\tau'-\mathsf{E}(p^{-1}\tau\langle 1+\Phi\rangle\,\Phi'+p^{-1}\langle 1+\Phi\rangle\,\Phi_{xy})-\mathsf{M}H_{y,y}\\
&\qquad\quad+\mathsf{EM}(p^{-1}\kappa\langle 1+\Phi\rangle\,\Phi_y+p^{-3}\Phi_x\Phi_y)\}\,y=0,\\
&\frac{d}{dz}\left(p\frac{dy}{dz}\right)-(\mathsf{M}H_z-2p\tau)\frac{dx}{dz}\\
&\qquad+\{p\tau'+\mathsf{E}(p^{-1}\tau\langle 1+\Phi\rangle\,\Phi'-p^{-1}\langle 1+\Phi\rangle\,\Phi_{xy})-\mathsf{M}(H_z'+H_{y,y})\\
&\qquad\quad+\mathsf{EM}(p^{-1}\kappa\langle 1+\Phi\rangle\,\Phi_y+p^{-3}\Phi_x\Phi_y)\}\,x\\
&\qquad+\{-p\tau^2-(1-\mathsf{M})\,p\kappa^2+\mathsf{E}(p^{-1}\langle 1+\Phi\rangle\,\Phi''+p^{-1}\langle 1+\Phi\rangle\,\Phi_{xx})\\
&\qquad\quad+\mathsf{M}(\tau H_z+H_{y,x})+\mathsf{EM}(-p^{-1}\kappa\langle 1+\Phi\rangle\,\Phi_x+p^{-3}\Phi_y^2)\}\,y=0.
\end{aligned}\right\}\quad(5\cdot 5)$$

It is seen that each of the ray equations involves, in general, both x and y; in consequence, the expression of the general solution of (5·5) necessitates the introduction of eight functions of z. On considering the point characteristic function which represents the paraxial imaging properties of the system, it is found that the condition that two planes should be stigmatically imaged is the vanishing of a certain set of three coefficients; the imaging is geometrically similar only if two relations are satisfied among another set of three coefficients.

The treatment of the paraxial properties and, consequently, the subsequent calculation of aberrations also, are simplified considerably if we limit ourselves to systems for which we may so transform the x-, y-axes that each of the equations (5·5) contains only one off-axis co-ordinate. Such systems will be termed 'orthogonal'.

6. The orthogonality condition

It is well known that it is often necessary to perform co-ordinate transformations in order to bring to light the paraxial image-forming properties of electromagnetic fields. In this section we shall investigate under what conditions such a transformation is profitable and consider its consequences.

It will be convenient to write the paraxial variational function (4·9) in the form

$$m^{(2)}=\tfrac{1}{2}p(x'^2+y'^2)+\tfrac{1}{2}t(x'y-xy')+\tfrac{1}{2}qx^2+\tfrac{1}{2}rxy+\tfrac{1}{2}sy^2,\quad(6\cdot 1)$$

where

$$\left.\begin{aligned}
t&=\mathsf{M}H_z-2p\tau,\\
q&=p\tau^2-(1-\mathsf{M})\,3p\langle 1+\tfrac{2}{3}p^2\rangle\kappa^2\langle 1+\Phi\rangle^{-2}+\mathsf{E}p^{-1}\langle 1+\Phi\rangle\,\Phi_{xx}\\
&\qquad+\mathsf{M}(-p\kappa^2-\tau H_z+H_{y,x})-\mathsf{EM}(p^{-1}\kappa\langle 1+\Phi\rangle\,\Phi_x+p^{-3}\Phi_x^2),\\
r&=\mathsf{E}2p^{-1}\langle 1+\Phi\rangle\,\Phi_{xy}+\mathsf{M}(H_z'+2H_{y,y})-\mathsf{EM}(2p^{-1}\kappa\langle 1+\Phi\rangle\,\Phi_y+2p^{-3}\Phi_x\Phi_y),\\
s&=p\tau^2+(1-\mathsf{M})\,p\kappa^2-\mathsf{E}(p^{-1}\langle 1+\Phi\rangle\,\Phi''+p^{-1}\langle 1+\Phi\rangle\,\Phi_{xx})\\
&\qquad-\mathsf{M}(\tau H_z+H_{y,x})+\mathsf{EM}(p^{-1}\kappa\langle 1+\Phi\rangle\,\Phi_x-p^{-3}\Phi_y^2).
\end{aligned}\right\}\quad(6\cdot 2)$$

Let us now suppose that it is possible to transform the co-ordinates (x, y) into co-ordinates (u, v) so that $m^{(2)}$ takes the form

$$m^{(2)}=\tfrac{1}{2}(pu'^2+Uu^2)+\tfrac{1}{2}(pv'^2+Vv^2).\quad(6\cdot 3)$$

The Lagrangian equations are then

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(p\frac{\mathrm{d}u}{\mathrm{d}z}\right)-Uu=0 \quad\text{and}\quad \frac{\mathrm{d}}{\mathrm{d}z}\left(p\frac{\mathrm{d}v}{\mathrm{d}z}\right)-Vv=0, \tag{6·4}$$

which are in the 'separated' or 'orthogonal' form which greatly simplifies the investigation of paraxial properties; if $U(z) = V(z)$, the paraxial system is 'Gaussian'. Let us therefore consider under what conditions it is possible to transform the function (6·1) into the form (6·3).

There are, fortunately, restrictions upon the transformations it is profitable to consider. If the separation can be effected at all, it can be done by a transformation which does not change the scale of the co-ordinates as is obvious, for, once the separation is achieved, magnification of the co-ordinates cannot create 'mixed' terms but it can restore the original scale. It is also seen that we should neither adopt oblique co-ordinates, for the transformation then introduces a term in $u'v'$, nor displace the origin, for first-order terms then appear.

It follows that the only transformation of interest is one which represents a rotation of the x-y plane about the ray-axis. Such a transformation may be written in the form

$$x+\mathrm{i}y = (u+\mathrm{i}v)\,\mathrm{e}^{\mathrm{i}\chi}, \tag{6·5}$$

where $\chi(z)$ is real.

By partial integration of the variational equation, we may verify that (6·3) is obtained from (6·1) by means of the transformation (6·5) if and only if

$$t = 2p\chi' \tag{6·6}$$

and

$$\left.\begin{aligned} r &= (U-V)\sin 2\chi,\\ q &= \tfrac{1}{2}(U+V)+\tfrac{1}{2}(U-V)\cos 2\chi+p\chi'^2,\\ s &= \tfrac{1}{2}(U+V)-\tfrac{1}{2}(U-V)\cos 2\chi+p\chi'^2.\end{aligned}\right\} \tag{6·7}$$

These may otherwise be regarded as a set of four equations to be satisfied by the three quantities U, V and χ; we should therefore establish the condition for the existence of a solution to these equations. This condition may be obtained by noting that the equations (6·7) give

$$\tan 2\chi = r/(q-s), \tag{6·8}$$

and that this is compatible with (6·6) if and only if

$$p\{r'(q-s)-r(q'-s')\} = t\{(q-s)^2+r^2\}. \tag{6·9}$$

The relation (6·9) *is* therefore *the 'orthogonality condition'*.

If the condition (6·9) is satisfied, we may solve (6·7) for U and V to obtain

$$\left.\begin{aligned} U &= \tfrac{1}{2}(q+s)+\tfrac{1}{2}\sqrt{((q-s)^2+r^2)}-\tfrac{1}{4}p^{-1}t^2,\\ V &= \tfrac{1}{2}(q+s)-\tfrac{1}{2}\sqrt{((q-s)^2+r^2)}-\tfrac{1}{4}p^{-1}t^2.\end{aligned}\right\} \tag{6·10}$$

Hence if r and $q-s$ do not both vanish and if (6·9) is satisfied, the functions (6·1) may be transformed to (6·3) by means of (6·5); the function χ is given by (6·8) and U and V by (6·10).

If r and $q-s$ both vanish, the orthogonality condition is satisfied but χ cannot be obtained from (6·8); it is then defined only by (6·6) which leaves it undetermined to the extent of an additive constant. We also find from (6·10) that U and V are then equal being given by

$$U = V = q-\tfrac{1}{4}p^{-1}t^2. \tag{6·11}$$

This is the Gaussian case which is met in the study of fields of rotational symmetry so that *the condition that the function* (6·1) *should be Gaussian is that* $r = 0$ *and* $q = s$.

It is proposed that the u-, v-axes, for which an orthogonal paraxial function takes on the form (6·3) should be termed the *principal axes* of the system. The principal axes are determined uniquely unless the system is Gaussian when they are undetermined to the extent of an arbitrary uniform rotation about the ray-axis.

The general solution of (6·4) may be written as

$$\left.\begin{aligned} u(z) &= u_o f(z) + u_a h(z), \\ v(z) &= v_o g(z) + v_a k(z), \end{aligned}\right\} \tag{6·12}$$

where (u_o, v_o) and (u_a, v_a) are the co-ordinates of the points of intersection of the ray with the object and aperture planes $z = z_o$ and $z = z_a$, respectively, and the functions $f(z)$, $g(z)$, $h(z)$ and $k(z)$ satisfy the boundary conditions

$$f_o = g_o = 1, \quad f_a = g_a = 0, \quad h_o = k_o = 0 \quad \text{and} \quad h_a = k_a = 1. \tag{6·13}$$

The quantities defined by

$$\left.\begin{aligned} Q_u &= p(f'h - fh'), \\ Q_v &= p(g'k - gk') \end{aligned}\right\} \tag{6·14}$$

are constants and are therefore given by

$$\left.\begin{aligned} Q_u &= -p_o h'_o \quad \text{or} \quad p_a f'_a, \\ Q_v &= -p_o k'_o \quad \text{or} \quad p_a g'_a. \end{aligned}\right\} \tag{6·15}$$

For the purposes of aberration theory, it is convenient to define four more functions by

$$\left.\begin{aligned} \alpha(z) &= Q_u^{-1} f(z), \qquad \gamma(z) = Q_u^{-1} h(z), \\ \beta(z) &= Q_v^{-1} g(z), \qquad \theta(z) = Q_v^{-1} k(z). \end{aligned}\right\} \tag{6·16}$$

It is clear from (6·12) that the condition that the object plane $z = z_o$ should be stigmatically imaged upon the plane $z = z_b$ is that

$$h_b = 0 \quad \text{and} \quad k_b = 0, \tag{6·17}$$

from which it follows also that

$$\gamma_b = 0 \quad \text{and} \quad \theta_b = 0. \tag{6·18}$$

The imaging will be isotropic if $f_b = g_b$, the magnification being given by either side of this equation.

It is necessary also to note that, since Q_u and Q_v are constant,

$$\left.\begin{aligned} \alpha'_b &= Q_u^{-1} f'_b, \quad \gamma'_b = Q_u^{-1} h'_b, \\ \beta'_b &= Q_v^{-1} g'_b, \quad \theta'_b = Q_v^{-1} k'_b. \end{aligned}\right\} \tag{6·19}$$

It is possible to eliminate the first-order derivatives from the ray equations (6·4) by means of Scherzer's transformation (Scherzer 1936)

$$u = p^{-\frac{1}{4}} \tilde{u}, \quad v = p^{-\frac{1}{4}} \tilde{v}; \tag{6·20}$$

the function p then disappears from the ray equations. It is particularly interesting to note that the new functions $\tilde{U}(z)$ and $\tilde{V}(z)$ then satisfy the relation

$$\tilde{U} + \tilde{V} = -\kappa^2 - \tfrac{1}{2} p^{-4} (3 + p^2)\, \Phi'^2 - p^{-4} \Phi_x^2 - p^{-4} \Phi_y^2 - \tfrac{1}{2} p^{-2} H_z^2, \tag{6·21}$$

from which it is clear that $\tilde{U}$ and $\tilde{V}$ cannot both be positive. We may now state that *if the optical properties are referred to the co-ordinate system* $(\tilde{u}, \tilde{v}, z)$, *any orthogonal system must be convergent in at least one of its principal directions*. If the field is purely magnetic, p is constant so that the statement holds also for the co-ordinate system (u, v, z).

7. The zero-order chromatic aberration

We saw in the last section that the function $m^{(2)}$ is responsible for the paraxial imaging properties of the system. We must now investigate the effect upon the image-formation of the functions given by (4·10), (4·11), (4·12), (4·15) and (4·16). These all produce aberrations which may be calculated by the method of perturbation characteristic functions as set out in a recent paper (Sturrock 1951); references to formulae in this paper will be denoted by the letters 'P.C.F.'. In applying these formulae to the present problem we must replace x_1, x_2 by u, v; n_1, n_2 by n_u, n_v; g_1, g_2, h_1, h_2 by f, g, h, k; k_1, k_2 by Q_u, Q_v; and, consequently, $k_1^{-1}g_1$, etc., by α, etc.

Let us consider the function $m^{(1)\mathrm{I}}$, given by (4·11), which is responsible for the zero-order chromatic aberration. We may write

$$m^{(1)\mathrm{I}} = A_1 x + A_2 y, \tag{7·1}$$

where, in view of (5·1),

$$\left.\begin{aligned} A_1 &= -(1-\mathsf{M})\,2p^{-1}\langle 1+\tfrac{1}{2}p^2\rangle\,\kappa\langle 1+\Phi\rangle^{-1} - \mathsf{M}p^{-1}\kappa\langle 1+\Phi\rangle - \mathsf{E}\mathsf{M}p^{-3}\Phi_x, \\ A_2 &= -\mathsf{E}\mathsf{M}p^{-3}\Phi_y. \end{aligned}\right\} \tag{7·2}$$

Calculation of the zero-order relativistic correction, due to the function $m^{(1)R}$, is identical with calculation of the zero-order relativistic correction. We need only replace (7·2) by

$$\left.\begin{aligned} A_1^* &= (1-\mathsf{M})\,\tfrac{1}{4}p^3\kappa + \mathsf{M}\tfrac{1}{8}(-p^3\kappa + 3p\Phi_x) \\ A_2^* &= \mathsf{M}\tfrac{3}{8}p\Phi_y \end{aligned}\right\} \quad (\mathsf{E}=1). \tag{7·3}$$

The relativistic correction will, of course, be applied only if the field is purely electric or combined electric and magnetic since there is no difference between relativistic and non-relativistic calculations for magnetic fields, apart from the relation of p to Φ.

Upon the transformation (6·5), (7·1) becomes

$$m^{(1)\mathrm{I}} = B_1 u + B_2 v, \tag{7·4}$$

where

$$\left.\begin{aligned} B_1 &= A_1 \cos\chi + A_2 \sin\chi, \\ B_2 &= -A_1 \sin\chi + A_2 \cos\chi. \end{aligned}\right\} \tag{7·5}$$

If, according to P.C.F. (7·9), we introduce the characteristic functions defined by

$$*V_{oc}^{(1)\mathrm{I}} = \int_{z_o}^{z_c} m^{(1)\mathrm{I}}\,\mathrm{d}z, \quad *V_{ac}^{(1)\mathrm{I}} = \int_{z_a}^{z_c} m^{(1)\mathrm{I}}\,\mathrm{d}z, \tag{7·6}$$

we find, with the help of (6·12), that

$$*V_{oc}^{(1)\mathrm{I}} = C_{1oc}u_o + C_{2oc}v_o + C_{3oc}u_a + C_{4oc}v_a, \tag{7·7}$$

where

$$\left.\begin{aligned} C_{1oc} &= \int_{z_o}^{z_c} B_1 f\,\mathrm{d}z, \quad C_{3oc} = \int_{z_o}^{z_c} B_1 h\,\mathrm{d}z, \\ C_{2oc} &= \int_{z_o}^{z_c} B_2 g\,\mathrm{d}z, \quad C_{4oc} = \int_{z_o}^{z_c} B_2 k\,\mathrm{d}z. \end{aligned}\right\} \tag{7·8}$$

The same formulae hold for the other characteristic function if the suffix o is replaced by a.

The formula P.C.F. (7·10) becomes, in the present notation,

$$\left.\begin{aligned} u_c^{(0)\mathrm{I}} &= \alpha_c \frac{\partial * V_{oc}^{(1)\mathrm{I}}}{\partial u_a} - \gamma_c \frac{\partial * V_{ac}^{(1)\mathrm{I}}}{\partial u_o}, \\ v_c^{(0)\mathrm{I}} &= \beta_c \frac{\partial * V_{oc}^{(1)\mathrm{I}}}{\partial v_a} - \theta_c \frac{\partial * V_{ac}^{(1)\mathrm{I}}}{\partial v_o} \end{aligned}\right\} \tag{7·9}$$

from which we find that $u^{(0)\mathrm{I}}(z) = D_1(z), \quad v^{(0)\mathrm{I}}(z) = D_2(z),$ (7·10)

where (with the usual notation $D_{1c} = D_1(z_c)$, etc.)

$$\left.\begin{aligned} D_{1c} &= \alpha_c C_{3oc} - \gamma_c C_{1ac}, \\ D_{2c} &= \beta_c C_{4oc} - \theta_c C_{2ac}. \end{aligned}\right\} \tag{7·11}$$

The functions $D_1(z)$, $D_2(z)$ are the coefficients of zero-order chromatic aberration (per unit increase of beam energy) along the ray-axis. By writing

$$u_b^{(0)\mathrm{I}} = E_1, \quad v_b^{(0)\mathrm{I}} = E_2, \tag{7·12}$$

we introduce E_1, E_2, the coefficients of zero-order chromatic aberration in the image plane. From P.C.F. (7·11) or from (7·11) and (6·18), we see that

$$E_1 = \alpha_b C_{3ob}, \quad E_2 = \beta_b C_{4ob}. \tag{7·13}$$

If we do not wish subsequently to calculate the paraxial chromatic aberration, it is not necessary to evaluate the zero-order chromatic aberration along the ray-axis but only in the image plane so that we need calculate only the integrals C_{3ob} and C_{4ob}.

If we are calculating not chromatic aberration but relativistic correction, (7·2) being replaced by (7·3), we arrive at coefficients $D_1^*(z)$, $D_2^*(z)$ and E_1^*, E_2^* which characterize the zero-order relativistic correction, (7·10) and (7·12) being replaced by

$$u^{(0)R}(z) = D_1^*(z), \quad v^{(0)R} = D_2^*(z) \tag{7·14}$$

and

$$u_b^{(0)R} = E_1^*, \quad v_b^{(0)R} = E_2^* \tag{7·15}$$

respectively.

It is clear that the coefficients E_1, E_2 characterize a pure displacement of the image, upon chromatic variation, of amount ϵE_1 in the u-direction and ϵE_2 in the v-direction.

8. The second-order aberration

If the beam is monochromatic, the imaging properties of the system approach the paraxial properties—as calculated in § 6—as the size of the object and the size of the aperture become indefinitely small. The function $m^{(3)}$, given by (4·10), gives rise to the principal discrepancy between the real image and the paraxial approximation to the image, the second-order aberration. This section will be devoted to a treatment of this aberration.

Let us suppose that, upon the transformation (6·5), (4·10) takes the form

$$\begin{aligned} m^{(3)} = {} & F_1 u(u'^2+v'^2)+F_2 v(u'^2+v'^2) \\ & +F_3 u(u'v-uv')+F_4 v(u'v-uv') \\ & +F_5 u^3+F_6 u^2 v+F_7 uv^2+F_8 v^3. \end{aligned} \tag{8·1}$$

Then we find, taking into account the conditions (5·4), that the functions $F_1(z)$, etc.—upon which calculation of the second-order aberrations will be based—may be derived from the formulae

$$\left.\begin{aligned} F_1 &= G_1\cos\chi+G_2\sin\chi, \\ F_2 &= -G_1\sin\chi+G_2\cos\chi, \\ F_3 &= G_3\cos\chi+G_4\sin\chi, \\ F_4 &= -G_3\sin\chi+G_4\cos\chi, \\ F_5 &= G_5\cos\chi+G_6\sin\chi+G_7\cos 3\chi+G_8\sin 3\chi, \\ F_6 &= -G_5\sin\chi+G_6\cos\chi-3G_7\sin 3\chi+3G_8\cos 3\chi, \\ F_7 &= G_5\cos\chi+G_6\sin\chi-3G_7\cos 3\chi-3G_8\sin 3\chi, \\ F_8 &= -G_5\sin\chi+G_6\cos\chi+G_7\sin 3\chi-G_8\cos 3\chi, \end{aligned}\right\} \tag{8·2}$$

where*

$$
\left.\begin{aligned}
G_1 &= (1-\mathsf{M})\,p\kappa+\mathsf{M}\tfrac{1}{2}p\kappa+\mathsf{EM}\tfrac{1}{2}p^{-1}\langle 1+\Phi\rangle\,\Phi_x,\\
G_2 &= \mathsf{EM}\tfrac{1}{2}p^{-1}\langle 1+\Phi\rangle\,\Phi_y,\\
G_3 &= -\mathsf{M}\tfrac{1}{6}(2p\kappa\tau+\kappa H_z)+\mathsf{EM}\tfrac{1}{6}(2p^{-1}\tau\langle 1+\Phi\rangle\,\Phi_x\\
&\quad +2p^{-1}\langle 1+\Phi\rangle\,\Phi'_y-2p^{-3}\Phi'\Phi_y-3p^{-2}\langle 1+\Phi\rangle\,\Phi_x H_z),\\
G_4 &= \mathsf{M}\tfrac{1}{3}p\kappa'+\mathsf{EM}\tfrac{1}{6}(2p^{-1}\kappa\langle 1+\Phi\rangle\,\Phi'-2p^{-1}\langle 1+\Phi\rangle\,\Phi'_x\\
&\quad +2p^{-3}\Phi'\Phi_x+2p^{-1}\tau\langle 1+\Phi\rangle\,\Phi_y-3p^{-2}\langle 1+\Phi\rangle\,\Phi_y H_z),\\
G_5 &= (1-\mathsf{M})\tfrac{1}{8}(-p\kappa''+p\kappa\tau^2+5p\kappa^3\langle 1+\Phi\rangle^{-2}-5p^{-1}\langle 1+\tfrac{3}{5}p^2\rangle\,\kappa'\langle 1+\Phi\rangle^{-1}\,\Phi'\\
&\quad +2p^{-1}\kappa(1-\langle 0\rangle)\,\langle 1+\Phi\rangle^{-2}\,\Phi'^2-2p^{-1}\kappa\langle 1+\Phi\rangle\,\Phi''-3p^{-1}\langle 1+\tfrac{1}{3}p^2\rangle\,\kappa\langle 1+\Phi\rangle^{-1}\,\Phi_{xx})\\
&\quad +\mathsf{M}\tfrac{1}{24}(-p\kappa''+p\kappa\tau^2+3\kappa\tau H_z-3\kappa H_{y,\,x}-p^{-1}\kappa H_z^2)\\
&\quad +\mathsf{EM}\tfrac{1}{24}(-5p^{-1}\kappa'\langle 1+\Phi\rangle^{-1}\,\Phi'+p^{-3}\kappa\Phi'^2-4p^{-1}\kappa\langle 1+\Phi\rangle\,\Phi''+2p^{-1}\tau^2\langle 1+\Phi\rangle\,\Phi_x\\
&\quad +3p^{-5}\langle 1+\Phi\rangle\,\Phi'^2\Phi_x+2p^{-3}\Phi''\Phi_x-2p^{-3}\Phi'\Phi'_x-2p^{-1}\langle 1+\Phi\rangle\,\Phi''_x+6p^{-3}\kappa\Phi_x^2\\
&\quad +9p^{-5}\langle 1+\Phi\rangle\,\Phi_x^3+2p^{-1}\tau'\langle 1+\Phi\rangle\,\Phi_y+2p^{-3}\tau\Phi'\Phi_y+4p^{-1}\tau\langle 1+\Phi\rangle\,\Phi'_y+3p^{-3}\kappa\Phi_y^2\\
&\quad +9p^{-5}\langle 1+\Phi\rangle\,\Phi_x\Phi_y^2-3p^{-1}\kappa\langle 1+\Phi\rangle\,\Phi_{xx}-6p^{-3}\Phi_x\Phi_{xx}-6p^{-3}\Phi_y\Phi_{xy}\\
&\quad -4p^{-2}\tau\langle 1+\Phi\rangle\,\Phi_x H_z+4p^{-4}\Phi'\Phi_y H_z-4p^{-2}\langle 1+\Phi\rangle\,\Phi'_y H_z+3p^{-3}\langle 1+\Phi\rangle\,\Phi_x H_z^2),\\
G_6 &= (1-\mathsf{M})\tfrac{1}{8}(-2p\kappa'\tau-p\kappa\tau'-5p^{-1}\langle 1+\tfrac{3}{5}p^2\rangle\,\kappa\tau\langle 1+\Phi\rangle^{-1}\,\Phi'-3p^{-1}\langle 1+\tfrac{1}{3}p^2\rangle\,\kappa\langle 1+\Phi\rangle^{-1}\,\Phi_{xy})\\
&\quad +\mathsf{M}\tfrac{1}{24}(-2p\kappa'\tau-p\kappa\tau'-3\kappa' H_z+\kappa H'_z-3\kappa H_{y,\,y})\\
&\quad +\mathsf{EM}\tfrac{1}{24}(-5p^{-1}\kappa\tau\langle 1+\Phi\rangle\,\Phi'-2p^{-1}\tau'\langle 1+\Phi\rangle\,\Phi_x-2p^{-3}\tau\Phi'\Phi_x-4p^{-1}\tau\langle 1+\Phi\rangle\,\Phi'_x\\
&\quad +2p^{-1}\tau^2\langle 1+\Phi\rangle\,\Phi_y+3p^{-5}\langle 1+\Phi\rangle\,\Phi'^2\Phi_y+8p^{-3}\Phi''\Phi_\gamma-2p^{-3}\Phi'\Phi'_y-2p^{-1}\langle 1+\Phi\rangle\,\Phi''_y\\
&\quad -3p^{-3}\kappa\Phi_x\Phi_y+9p^{-5}\langle 1+\Phi\rangle\,\Phi_x^2\Phi_y+9p^{-5}\langle 1+\Phi\rangle\,\Phi_y^3+6p^{-3}\Phi_y\Phi_{xx}-3p^{-1}\kappa\langle 1+\Phi\rangle\,\Phi_{xy}\\
&\quad -6p^{-3}\Phi_x\Phi_{xy}-4p^{-2}\kappa\langle 1+\Phi\rangle\,\Phi' H_z+4p^{-2}\langle 1+\Phi\rangle\,\Phi'_x H_z-4p^{-4}\Phi'\Phi_x H_z\\
&\quad -4p^{-2}\tau\langle 1+\Phi\rangle\,\Phi_y H_z+3p^{-3}\langle 1+\Phi\rangle\,\Phi_y H_z^2),\\
G_7 &= (1-\mathsf{M})\tfrac{1}{24}(3p\kappa''-3p\kappa\tau^2+9p\kappa^3\langle 1+\Phi\rangle^{-2}+15p^{-1}\langle 1+\tfrac{3}{5}p^2\rangle\,\kappa'\langle 1+\Phi\rangle^{-1}\,\Phi'\\
&\quad -6p^{-1}\kappa(1-\langle 0\rangle)\,\langle 1+\Phi\rangle^{-2}\,\Phi'^2+6p^{-1}\kappa\langle 1+\Phi\rangle\,\Phi''-15p^{-1}\langle 1+\tfrac{3}{5}p^2\rangle\,\kappa\langle 1+\Phi\rangle^{-1}\,\Phi_{xx}\\
&\quad +4p^{-1}\langle 1+\Phi\rangle\,\Phi_{xxx})+\mathsf{M}\tfrac{1}{24}(p\kappa''-p\kappa\tau^2+\kappa\tau H_z-5\kappa H_{y,\,x}+4H_{y,\,xx})\\
&\quad +\mathsf{EM}\tfrac{1}{24}(5p^{-1}\kappa'\langle 1+\Phi\rangle\,\Phi'+4p^{-1}\kappa\langle 1+\Phi\rangle\,\Phi''-p^{-3}\kappa\Phi'^2-2p^{-1}\tau^2\langle 1+\Phi\rangle\,\Phi_x\\
&\quad -3p^{-5}\langle 1+\Phi\rangle\,\Phi'^2\Phi_x-2p^{-3}\Phi''\Phi_x+2p^{-3}\Phi'\Phi'_x+2p^{-1}\langle 1+\Phi\rangle\,\Phi''_x+6p^{-3}\kappa\Phi_x^2\\
&\quad +3p^{-5}\langle 1+\Phi\rangle\,\Phi_x^3-2p^{-1}\tau'\langle 1+\Phi\rangle\,\Phi_y-2p^{-3}\tau\Phi'\Phi_y-4p^{-1}\tau\langle 1+\Phi\rangle\,\Phi'_y-3p^{-3}\kappa\Phi_y^2\\
&\quad -9p^{-5}\langle 1+\Phi\rangle\,\Phi_x\Phi_y^2-9p^{-1}\kappa\langle 1+\Phi\rangle\,\Phi_{xx}-6p^{-3}\Phi_x\Phi_{xx}+6p^{-3}\Phi_y\Phi_{xy}+4p^{-1}\langle 1+\Phi\rangle\,\Phi_{xxx}),\\
G_8 &= (1-\mathsf{M})\tfrac{1}{24}(2p\kappa'\tau+p\kappa\tau'+5p^{-1}\langle 1+\tfrac{3}{5}p^2\rangle\,\kappa\tau\langle 1+\Phi\rangle^{-1}\,\Phi'-13p^{-1}\langle 1+\tfrac{7}{13}p^2\rangle\,\kappa\langle 1+\Phi\rangle^{-1}\,\Phi_{xy}\\
&\quad +4p^{-1}\langle 1+\Phi\rangle\,\Phi_{xxy})+\mathsf{M}\tfrac{1}{24}(-2p\kappa'\tau-p\kappa\tau'+\kappa' H_z+\kappa H'_z-3\kappa H_{y,\,y}+4H_{y,\,xy})\\
&\quad +\mathsf{EM}\tfrac{1}{24}(-p^{-1}\kappa\tau\langle 1+\Phi\rangle\,\Phi'+2p^{-1}\tau'\langle 1+\Phi\rangle\,\Phi_x-2p^{-3}\tau\Phi'\Phi_x+4p^{-1}\tau\langle 1+\Phi\rangle\,\Phi'_x\\
&\quad -2p^{-1}\tau^2\langle 1+\Phi\rangle\,\Phi_y+3p^{-5}\langle 1+\Phi\rangle\,\Phi'^2\Phi_y-4p^{-3}\Phi''\Phi_y-2p^{-3}\Phi'\Phi'_y+2p^{-1}\langle 1+\Phi\rangle\,\Phi''_y\\
&\quad +9p^{-3}\kappa\Phi_x\Phi_y+9p^{-5}\langle 1+\Phi\rangle\,\Phi_x^2\Phi_y-3p^{-5}\langle 1+\Phi\rangle\,\Phi_y^3-6p^{-3}\Phi_y\Phi_{xx}\\
&\quad -7p^{-1}\kappa\langle 1+\Phi\rangle\,\Phi_{xy}-6p^{-3}\Phi_x\Phi_{xy}+4p^{-1}\langle 1+\Phi\rangle\,\Phi_{xxy}).
\end{aligned}\right\}\quad (8\cdot 3)
$$

* Note that $1-\langle 0\rangle = 1$ in relativistic calculations,
$= 0$ in non-relativistic calculations.

It should be remembered, at this point, that it is not suggested that it would be practicable to compute the properties of an arbitrary electron-optical system. Formulae are given in their general form only in order that any particular case of interest may be extracted. The formulae (8·3) simplify considerably if the field is purely electric, purely magnetic, or mirror-symmetrical, or if the field is uniform along the ray-axis and the latter is a helix.

Following P.C.F. (7·2), we shall introduce the characteristic functions defined by

$$*V_{oc}^{(3)} = \int_{z_o}^{z_c} m^{(3)}\,\mathrm{d}z, \quad *V_{ac}^{(3)} = \int_{z_a}^{z_c} m^{(3)}\,\mathrm{d}z. \tag{8·4}$$

We then find, from (6·12) and (8·1), that

$$\begin{aligned} *V_{oc}^{(3)} = {} & K_{1oc}u_o^3 + K_{2oc}u_o^2v_o + K_{3oc}u_o^2u_a + K_{4oc}u_o^2v_a + K_{5oc}u_ov_o^2 \\ & + K_{6oc}u_ov_ou_a + K_{7oc}u_ov_ov_a + K_{8oc}u_ou_a^2 + K_{9oc}u_ou_av_a + K_{10oc}u_ov_a^2 \\ & + K_{11oc}v_o^3 + K_{12oc}v_o^2u_a + K_{13oc}v_o^2v_a + K_{14oc}v_ou_a^2 + K_{15oc}v_ou_av_a \\ & + K_{16oc}v_ov_a^2 + K_{17oc}u_a^3 + K_{18oc}u_a^2v_a + K_{19oc}u_av_a^2 + K_{20oc}v_a^3, \end{aligned} \tag{8·5}$$

where

$$\left.\begin{aligned} K_{1oc} &= \int_{z_o}^{z_c} \{F_1ff'^2 + F_5f^3\}\,\mathrm{d}z, \\ K_{2oc} &= \int_{z_o}^{z_c} \{F_2gf'^2 + F_3f(f'g - fg') + F_6f^2g\}\,\mathrm{d}z, \\ K_{3oc} &= \int_{z_o}^{z_c} \{F_1(hf'^2 + 2ff'h') + 3F_5f^2h\}\,\mathrm{d}z, \\ K_{4oc} &= \int_{z_o}^{z_c} \{F_2kf'^2 + F_3f(f'k - fk') + F_6f^2k\}\,\mathrm{d}z, \\ K_{5oc} &= \int_{z_o}^{z_c} \{F_1fg'^2 + F_4g(f'g - fg') + F_7fg^2\}\,\mathrm{d}z, \\ K_{6oc} &= \int_{z_o}^{z_c} \{2F_2gf'h' - F_3f(g'h - gh') + F_3h(f'g - fg') + 2F_6fgh\}\,\mathrm{d}z, \\ K_{7oc} &= \int_{z_o}^{z_c} \{2F_1fg'k' + F_4g(f'k - fk') + F_4k(f'g - fg') + 2F_7fgk\}\,\mathrm{d}z, \\ K_{8oc} &= \int_{z_o}^{z_c} \{F_1(fh'^2 + 2hf'h') + 3F_5fh^2\}\,\mathrm{d}z, \\ K_{9oc} &= \int_{z_o}^{z_c} \{2F_2kf'h' + F_3f(h'k - hk') + F_3h(f'k - fk') + 2F_6fhk\}\,\mathrm{d}z, \\ K_{10oc} &= \int_{z_o}^{z_c} \{F_1fk'^2 + F_4k(f'k - fk') + F_7fk^2\}\,\mathrm{d}z, \\ K_{11oc} &= \int_{z_o}^{z_c} \{F_2gg'^2 + F_8g^3\}\,\mathrm{d}z, \\ K_{12oc} &= \int_{z_o}^{z_c} \{F_1hg'^2 - F_4g(g'h - gh') + F_7g^2h\}\,\mathrm{d}z, \\ K_{13oc} &= \int_{z_o}^{z_c} \{F_2(kg'^2 + 2gg'k') + 3F_8g^2k\}\,\mathrm{d}z, \\ K_{14oc} &= \int_{z_o}^{z_c} \{F_2gh'^2 - F_3h(g'h - gh') + F_6gh^2\}\,\mathrm{d}z, \end{aligned}\right\} \tag{8·6}$$

$$\left.\begin{aligned}
K_{15oc} &= \int_{z_o}^{z_c} \{2F_1 hg'k' + F_4 g(h'k - hk') - F_4 k(g'h - gh') + 2F_7 ghk\}\,dz,\\
K_{16oc} &= \int_{z_o}^{z_c} \{F_2(gk'^2 + 2kg'k') + 3F_8 gk^2\}\,dz,\\
K_{17oc} &= \int_{z_o}^{z_c} \{F_1 hh'^2 + F_5 h^3\}\,dz,\\
K_{18oc} &= \int_{z_o}^{z_c} \{F_2 kh'^2 + F_3 h(h'k - hk') + F_6 h^2 k\}\,dz,\\
K_{19oc} &= \int_{z_o}^{z_c} \{F_1 hk'^2 + F_4 k(h'k - hk') + F_7 hk^2\}\,dz,\\
K_{20oc} &= \int_{z_o}^{z_c} \{F_2 kk'^2 + F_8 k^3\}\,dz.
\end{aligned}\right\}$$

The same formulae hold for the other characteristic function if the suffix o is replaced by a, but the coefficients K_{1ac}, etc., may also be evaluated from the coefficients K_{1oc}, etc., by means of the relations

$$K_{1ac} = K_{1oc} - K_{1oa}, \text{ etc.} \tag{8·7}$$

The equation P.C.F. (7·3) becomes, in the present notation,

$$\left.\begin{aligned}
u_c^{(2)} &= \alpha_c \frac{\partial^* V_{oc}^{(3)}}{\partial u_a} - \gamma_c \frac{\partial^* V_{ac}^{(3)}}{\partial u_o},\\
v_c^{(2)} &= \beta_c \frac{\partial^* V_{oc}^{(3)}}{\partial v_a} - \theta_c \frac{\partial^* V_{ac}^{(3)}}{\partial v_o},
\end{aligned}\right\} \tag{8·8}$$

from which we obtain the following expression for the second-order aberration along the ray-axis:

$$\left.\begin{aligned}
u^{(2)}(z) = {} & L_1(z)\,u_o^2 + L_2(z)\,u_o v_o + L_3(z)\,u_o u_a + L_4(z)\,u_o v_a\\
& + L_5(z)\,v_o^2 + L_6(z)\,v_o u_a + L_7(z)\,v_o v_a\\
& + L_8(z)\,u_a^2 + L_9(z)\,u_a v_a + L_{10}(z)\,v_a^2,\\
v^{(2)}(z) = {} & L_{11}(z)\,u_o^2 + L_{12}(z)\,u_o v_o + L_{13}(z)\,u_o u_a + L_{14}(z)\,u_o v_a\\
& + L_{15}(z)\,v_o^2 + L_{16}(z)\,v_o u_a + L_{17}(z)\,v_o v_a\\
& + L_{18}(z)\,u_a^2 + L_{19}(z)\,u_a v_a + L_{20}(z)\,v_a^2,
\end{aligned}\right\} \tag{8·9}$$

where

$$\left.\begin{aligned}
L_{1c} &= \alpha_c K_{3oc} - 3\gamma_c K_{1ac}, & L_{11c} &= \beta_c K_{4oc} - \theta_c K_{2ac},\\
L_{2c} &= \alpha_c K_{6oc} - 2\gamma_c K_{2ac}, & L_{12c} &= \beta_c K_{7oc} - 2\theta_c K_{5ac},\\
L_{3c} &= 2\alpha_c K_{8oc} - 2\gamma_c K_{3ac}, & L_{13c} &= \beta_c K_{9oc} - \theta_c K_{6ac},\\
L_{4c} &= \alpha_c K_{9oc} - 2\gamma_c K_{4ac}, & L_{14c} &= 2\beta_c K_{10oc} - \theta_c K_{7ac},\\
L_{5c} &= \alpha_c K_{12oc} - \gamma_c K_{5ac}, & L_{15c} &= \beta_c K_{13oc} - 3\theta_c K_{11ac},\\
L_{6c} &= 2\alpha_c K_{14oc} - \gamma_c K_{6ac}, & L_{16c} &= \beta_c K_{15oc} - 2\theta_c K_{12ac},\\
L_{7c} &= \alpha_c K_{15oc} - \gamma_c K_{7ac}, & L_{17c} &= 2\beta_c K_{16oc} - 2\theta_c K_{13ac},\\
L_{8c} &= 3\alpha_c K_{17oc} - \gamma_c K_{8ac}, & L_{18c} &= \beta_c K_{18oc} - \theta_c K_{14ac},\\
L_{9c} &= 2\alpha_c K_{18oc} - \gamma_c K_{9ac}, & L_{19c} &= 2\beta_c K_{19oc} - \theta_c K_{15ac},\\
L_{10c} &= \alpha_c K_{19oc} - \gamma_c K_{10ac}, & L_{20c} &= 3\beta_c K_{20oc} - \theta_c K_{16ac}.
\end{aligned}\right\} \tag{8·10}$$

The functions $L_1(z), \ldots, L_{20}(z)$ are the coefficients of the second-order aberration along the ray-axis.

The second-order aberration in the image plane may be written as

$$\left.\begin{aligned} u_b^{(2)} &= M_1 u_o^2 + M_2 u_o v_o + M_3 u_o u_a + M_4 u_o v_a \\ &\quad + M_5 v_o^2 + M_6 v_o u_a + M_7 v_o v_a \\ &\quad + M_8 u_a^2 + M_9 u_a v_a + M_{10} v_a^2, \\ v_b^{(2)} &= M_{11} u_o^2 + M_{12} u_o v_o + M_{13} u_o u_a + M_{14} u_o v_a \\ &\quad + M_{15} v_o^2 + M_{16} v_o u_a + M_{17} v_o v_a \\ &\quad + M_{18} u_a^2 + M_{19} u_a v_a + M_{20} v_a^2. \end{aligned}\right\} \tag{8·11}$$

We then find, either from P.C.F. (7·5) or from (8·10), that the coefficients of the second-order aberration in the image plane are given by

$$\left.\begin{array}{llll} M_1 = \alpha_b K_{3ob}, & M_6 = 2\alpha_b K_{14ob}, & M_{11} = \beta_b K_{4ob}, & M_{16} = \beta_b K_{15ob}, \\ M_2 = \alpha_b K_{6ob}, & M_7 = \alpha_b K_{15ob}, & M_{12} = \beta_b K_{7ob}, & M_{17} = 2\beta_b K_{16ob}, \\ M_3 = 2\alpha_b K_{8ob}, & M_8 = 3\alpha_b K_{17ob}, & M_{13} = \beta_b K_{9ob}, & M_{18} = \beta_b K_{18ob}, \\ M_4 = \alpha_b K_{9ob}, & M_9 = 2\alpha_b K_{18ob}, & M_{14} = 2\beta_b K_{10ob}, & M_{19} = 2\beta_b K_{19ob}, \\ M_5 = \alpha_b K_{12ob}, & M_{10} = \alpha_b K_{19ob}, & M_{15} = \beta_b K_{13ob}, & M_{20} = 3\beta_b K_{20ob}. \end{array}\right\} \tag{8·12}$$

We see from (8·12) that there are certain relations between the coefficients M_1, etc.; for instance, $M_{10}/\alpha_b = \frac{1}{2}M_{19}/\beta_b = K_{19ob}$. In order to obtain the geometrical forms of the second-order aberrations, we should therefore consider the separate effects of the terms K_{3ob}, etc., rather than the terms M_1, etc. We should find, for example, that the terms K_{18ob} and K_{19ob} are responsible for asymmetrical aberrations resembling coma except that the patterns are not pointed. The coefficients M_1, M_2, M_5, M_{11}, M_{12} and M_{15} represent pure distortions of the image; the coefficients $M_3, M_4, M_6, M_7, M_{13}, M_{14}, M_{16}$ and M_{17} characterize aberrations which depend on both the object co-ordinates and the aperture co-ordinates; and M_8, M_9, M_{10}, M_{18}, M_{19} and M_{20} characterize 'aperture' aberrations which are the same for all object points. The last set is generally the most important.

If it is not required subsequently to evaluate the paraxial chromatic aberration, it is not necessary to calculate the coefficients of (8·9) so as to find the second-order aberration for all z, but only to find the coefficients of (8·11) which gives the aberration in the image plane. In this case we need evaluate only the characteristic function $*V_{ob}^{(3)}$ which, in combination with P.C.F. (7·5), gives the formulae (8·12). It should be noted that it is now not necessary to evaluate the integrals K_{1ob}, K_{2ob}, K_{5ob} and K_{11ob}.

However, if the paraxial chromatic aberration will be required we shall need, in addition to $u^{(2)}(z)$, $v^{(2)}(z)$, the second-order aberration of the ray variables in the image plane. The relevant formulae, P.C.F. (7·4), become

$$\left.\begin{aligned} n_{ub}^{(2)} &= p_b \alpha_b' \frac{\partial {}^*V_{ob}^{(3)}}{\partial u_a} - p_b \gamma_b' \frac{\partial {}^*V_{ab}^{(3)}}{\partial u_o}, \\ n_{vb}^{(2)} &= p_b \beta_b' \frac{\partial {}^*V_{ob}^{(3)}}{\partial v_a} - p_b \theta_b' \frac{\partial {}^*V_{ab}^{(3)}}{\partial v_o}, \end{aligned}\right\} \tag{8·13}$$

from which we obtain the expressions

$$\left.\begin{aligned} n_{ub}^{(2)} = {} & N_1 u_o^2 + N_2 u_o v_o + N_3 u_o u_a + N_4 u_o v_a \\ & + N_5 v_o^2 + N_6 v_o u_a + N_7 v_o v_a \\ & + N_8 u_a^2 + N_9 u_a v_a + N_{10} v_a^2, \\ n_{vb}^{(2)} = {} & N_{11} u_o^2 + N_{12} u_o v_o + N_{13} u_o u_a + N_{14} u_o v_a \\ & + N_{15} v_o^2 + N_{16} v_o u_a + N_{17} v_o v_a \\ & + N_{18} u_a^2 + N_{19} u_a v_a + N_{20} v_a^2. \end{aligned}\right\} \qquad (8\cdot 14)$$

It is clear from a comparison of (8·8) and (8·13) that the coefficients N_1, etc., of (8·14) may be obtained from the formulae (8·10) for L_{1c}, etc., by replacing the suffix c by b, and $\alpha, \beta, \gamma, \theta$ by $p\alpha'$, $p\beta'$, $p\gamma'$, $p\theta'$, respectively.

Before we leave this section, let us investigate to what extent the coefficients of the second-order aberration in the image plane, $M_1, M_2, \ldots, M_{20}$, are arbitrary once the system is prescribed to the paraxial approximation. On referring to (8·3), we see that the field coefficients Φ_{xxx}, Φ_{xxy}, $H_{y,\,xx}$ and $H_{y,\,xy}$ are the only quantities appearing in the second-order formulae which are not present also in the treatment of the paraxial properties. We should therefore examine to what extent we may determine the coefficients of the second-order aberration by adjustment of these field coefficients.

These coefficients appear in G_7 and G_8 and hence in F_5, F_6, F_7 and F_8. We see from (8·6) that none of the latter is multiplied by the same combination of f, g, h and k in two different formulae. It would therefore appear that, if f, g, h and k are all different, we may adjust the coefficients Φ_{xxx}, etc., so as to give prescribed values to all the integrals of (8·6).

This possibility is not always realized, as can be seen by considering an important special case. We shall suppose that κ, τ and the paraxial field coefficients are independent of z; f, g, h and k are then circular functions. The coefficients Φ_{xxx}, etc., are therefore multiplied by sine and cosine functions; moreover, it is a consequence of the focusing condition (6·17) that the periods of the trigonometrical terms are commensurable. The four field coefficients will therefore appear in the integrals multiplied by certain members of a complete orthogonal set of functions. It follows that the number of integrals which we can prescribe cannot exceed twice the number of different orthogonal functions appearing in the set of terms $f^3, f^2g, \ldots, k^3$, for we have only two variable functions, $p^{-1}\langle 1+\Phi\rangle \Phi_{xxx} + H_{y,\,xx}$ and $p^{-1}\langle 1+\Phi\rangle \Phi_{xxy} + H_{y,\,xy}$.

Let us now suppose that the paraxial variational function is Gaussian; so that $f = g$ and $h = k$. Investigation of (8·6) then shows that half of the terms of the form $(f'g - fg')$ vanish while the other half are given by (6·14). It is also found that the following combinations of integrals do not involve the field coefficients Φ_{xxx}, etc.:

$$\left.\begin{matrix} 3K_{1ob} + K_{5ob}, & 2K_{4ob} - K_{6ob}, & K_{9ob} - 2K_{14ob}, \\ K_{2ob} + 3K_{11ob}, & K_{7ob} - 2K_{12ob}, & 2K_{10ob} - K_{15ob}, \\ 2K_{3ob} + K_{7ob}, & K_{8ob} + K_{10ob}, & 3K_{17ob} + K_{19ob}, \\ K_{4ob} + K_{13ob}, & K_{9ob} + 2K_{16ob}, & K_{18ob} + 2K_{20ob}. \end{matrix}\right\}$$

These twelve relations lead us to expect that we can prescribe only eight of the integrals (8·6); on considering only the integrals which contribute to the aberration coefficients of (8·12); this number is reduced to six. Hence *if the paraxial variational function is determined and is Gaussian, the second-order aberrations have only six 'degrees of freedom'*. We may verify that this number is not reduced if the paraxial field coefficients are independent of z for six orthogonal trigonometrical functions then appear.

9. The paraxial chromatic aberration

In the last two sections we have solved aberration problems of a type familiar in electron optics. The zero-order chromatic aberration and the second-order aberration, which we have calculated by means of first-order perturbation characteristic functions, may also be evaluated by the method of variation of parameters. In this section, however, we shall treat a problem which is not susceptible to the customary form of the latter method but which may be solved by the use of a second-order perturbation characteristic function.

We are to consider the influence on image-formation of the function $m^{(2)\mathrm{I}}$, given by (4·12). It will be found that this function is responsible for the paraxial terms of the chromatic aberration or, equivalently, the chromatic variation of the paraxial properties. Whereas the zero-order chromatic aberration is manifested as a *shift* of the image, in accordance with (7·12), the *paraxial* chromatic aberration exhibits itself as a change of magnification and a possible rotation, together with a defocusing, of the image.

Let us suppose that, upon the transformation (6·5), (4·12) takes the form

$$m^{(2)\mathrm{I}} = P_1(u'^2+v'^2)+P_2(u'v-uv')+P_3u^2+P_4uv+P_5v^2. \tag{9·1}$$

Then the functions $P_1(z)$, etc., are given by

$$\left.\begin{aligned} P_1 &= R_1,\\ P_2 &= R_2,\\ P_3 &= R_3+\ \ R_4\cos 2\chi+\ \ R_5\sin 2\chi,\\ P_4 &= \quad -2R_4\sin 2\chi+2R_5\cos 2\chi,\\ P_5 &= R_3-\ \ R_4\cos 2\chi-\ \ R_5\sin 2\chi, \end{aligned}\right\} \tag{9·2}$$

where

$$\left.\begin{aligned} R_1 &= \tfrac{1}{2}p^{-1}\langle 1+\Phi\rangle,\\ R_2 &= -\mathsf{M}\tfrac{1}{2}p^{-2}\langle 1+\Phi\rangle H_z,\\ R_3 &= (1-\mathsf{M})p^{-1}\kappa^2\langle 1+\Phi\rangle^{-1}+\mathsf{E}\tfrac{1}{4}p^{-3}\Phi''+\mathsf{M}\tfrac{1}{8}p^{-3}\langle 1+\Phi\rangle H_z^2\\ &\quad +\mathsf{EM}\tfrac{1}{4}(p^{-3}\kappa\Phi_x+3p^{-5}\langle 1+\Phi\rangle\Phi_x^2+3p^{-5}\langle 1+\Phi\rangle\Phi_y^2),\\ R_4 &= (1-\mathsf{M})\tfrac{3}{2}p^{-1}\kappa^2\langle 1+\Phi\rangle^{-1}-\mathsf{E}\tfrac{1}{4}(p^{-3}\Phi''+2p^{-3}\Phi_{xx})\\ &\quad +\mathsf{EM}\tfrac{3}{4}(p^{-3}\kappa\Phi_x+p^{-5}\langle 1+\Phi\rangle\Phi_x^2-p^{-5}\langle 1+\Phi\rangle\Phi_y^2),\\ R_5 &= -\mathsf{E}\tfrac{1}{2}p^{-3}\Phi_{xy}+\mathsf{EM}\tfrac{1}{2}(p^{-3}\kappa\Phi_y+3p^{-5}\langle 1+\Phi\rangle\Phi_x\Phi_y). \end{aligned}\right\} \tag{9·3}$$

As in § 7, the present work on chromatic aberration applies also to the relativistic correction. In order to obtain the paraxial relativistic correction, we replace R_1, etc., by R_1^*, etc., given by

$$\left.\begin{aligned} R_1^* &= \tfrac{1}{16}p^3, \\ R_2^* &= -\mathsf{M}\tfrac{1}{16}p^2H_z, \\ R_3^* &= -\tfrac{3}{32}p\Phi'' + \mathsf{M}\tfrac{1}{64}(-6p\kappa\Phi_x + 6p^{-1}\Phi_x^2 + 6p^{-1}\Phi_y^2 + pH_z^2), \\ R_4^* &= \tfrac{3}{32}(p\Phi'' + 2p\Phi_{xx}) - (1-\mathsf{M})\tfrac{3}{16}p^3\kappa^2 + \mathsf{M}\tfrac{3}{32}(-3p\kappa\Phi_x + p^{-1}\Phi_x^2 - p^{-1}\Phi_y^2), \\ R_5^* &= \tfrac{3}{16}p\Phi_{xy} + \tfrac{3}{16}\mathsf{M}(-p\kappa\Phi_y + p^{-1}\Phi_x\Phi_y). \end{aligned}\right\} \quad (\mathsf{E}=1) \qquad (9\cdot4)$$

These formulae are derived from (4·16).

Following P.C.F. (7·12), we now introduce the characteristic function defined by

$$*V_{ob}^{(2)\mathrm{I}} = \int_{z_o}^{z_b} \{m^{(2)\mathrm{I}} - \mathsf{D}^{(0)\mathrm{I}}\mathsf{D}^{(2)}m^{(2)}\}\,\mathrm{d}z, \qquad (9\cdot5)$$

where the operators $\mathsf{D}^{(0)\mathrm{I}}$ and $\mathsf{D}^{(2)}$ may be derived from the formula

$$\mathsf{D}^{\mathrm{I}} = u^{\mathrm{I}}\frac{\partial}{\partial u} + v^{\mathrm{I}}\frac{\partial}{\partial v} + u'^{\mathrm{I}}\frac{\partial}{\partial u'} + v'^{\mathrm{I}}\frac{\partial}{\partial v'}. \qquad (9\cdot6)$$

On evaluating (9·5) by means of (6·12) and (9·1), and (7·10), (8·11) and (6·3), we find that

$$\begin{aligned} *V_{ob}^{(2)\mathrm{I}} = {} & S_1u_o^2 + S_2u_ov_o + S_3u_ou_a + S_4u_ov_a \\ & + S_5v_o \quad + S_6v_ou_a + S_7v_ov_a \\ & + S_8u_a^2 \quad + S_9u_av_a + S_{10}v_a^2, \end{aligned} \qquad (9\cdot7)$$

where, in particular,

$$\left.\begin{aligned} S_3 &= \int_{z_o}^{z_b}\{2P_1f'h' + 2P_3fh - pD_1'L_3' - UD_1L_3 - pD_2'L_{13}' - VD_2L_{13}\}\,\mathrm{d}z, \\ S_4 &= \int_{z_o}^{z_b}\{P_2(f'k - fk') + P_4fk - pD_1'L_4' - UD_1L_4 - pD_2'L_{14}' - VD_2L_{14}\}\,\mathrm{d}z, \\ S_6 &= \int_{z_o}^{z_b}\{-P_2(g'h - gh') + P_4gh - pD_1'L_6' - UD_1L_6 - pD_2'L_{16}' - VD_2L_{16}\}\,\mathrm{d}z, \\ S_7 &= \int_{z_o}^{z_b}\{2P_1g'k' - 2P_5gk - pD_1'L_7' - UD_1L_7 - pD_2'L_{17}' - VD_2L_{17}\}\,\mathrm{d}z, \\ S_8 &= \int_{z_o}^{z_b}\{P_1h'^2 + P_3h^2 - pD_1'L_8' - UD_1L_8 - pD_2'L_{18}' - VD_2L_{18}\}\,\mathrm{d}z, \\ S_9 &= \int_{z_o}^{z_b}\{P_2(h'k - hk') + P_4hk - pD_1'L_9' - UD_1L_9 - pD_2'L_{19}' - VD_2L_{19}\}\,\mathrm{d}z, \\ S_{10} &= \int_{z_o}^{z_b}\{P_1k'^2 + P_5k^2 - pD_1'L_{10}' - UD_1L_{10} - pD_2'L_{20}' - VD_2L_{20}\}\,\mathrm{d}z; \end{aligned}\right\} \qquad (9\cdot8)$$

S_1, S_2 and S_5 will not be required.

The formulae P.C.F. (7·13) become, in the present notation,

$$\left.\begin{aligned} u_b^{(1)\mathrm{I}} &= \alpha_b\left\{\frac{\partial^* V_{ob}^{(2)\mathrm{I}}}{\partial u_a}+u_b^{(0)\mathrm{I}}\frac{\partial n_{ub}^{(2)}}{\partial u_a}+v_b^{(0)\mathrm{I}}\frac{\partial n_{vb}^{(2)}}{\partial u_a},\right. \\ v_b^{(1)\mathrm{I}} &= \beta_b\left.\left\{\frac{\partial^* V_{ob}^{(2)\mathrm{I}}}{\partial v_a}+u_b^{(0)\mathrm{I}}\frac{\partial n_{ub}^{(2)}}{\partial v_a}+v_b^{(0)\mathrm{I}}\frac{\partial n_{vb}^{(2)}}{\partial v_a}\right\}.\right.\end{aligned}\right\} \tag{9·9}$$

From these formulae we find, with the help of (7·12) and (8·14), that

$$\left.\begin{aligned} u_b^{(1)\mathrm{I}} &= T_1 u_o + T_2 v_o + T_3 u_a + T_4 v_a, \\ v_b^{(1)\mathrm{I}} &= T_5 u_o + T_6 v_o + T_7 u_a + T_8 v_a,\end{aligned}\right\} \tag{9·10}$$

where

$$\left.\begin{aligned} T_1 &= \alpha_b\{S_3 + E_1 N_3 + E_2 N_{13}\}, \\ T_2 &= \alpha_b\{S_6 + E_1 N_6 + E_2 N_{16}\}, \\ T_3 &= 2\alpha_b\{S_8 + E_1 N_8 + E_2 N_{18}\}, \\ T_4 &= \alpha_b\{S_9 + E_1 N_9 + E_2 N_{19}\}, \\ T_5 &= \beta_b\{S_4 + E_1 N_4 + E_2 N_{14}\}, \\ T_6 &= \beta_b\{S_7 + E_1 N_7 + E_2 N_{17}\}, \\ T_7 &= \beta_b\{S_9 + E_1 N_9 + E_2 N_{19}\}, \\ T_8 &= 2\beta_b\{S_{10} + E_1 N_{10} + E_2 N_{20}\}.\end{aligned}\right\} \tag{9·11}$$

It is seen that the values of N_1, N_2, N_5, N_{11}, N_{12} and N_{15} are not required. $T_1, \ldots, T_8$ are the coefficients of the paraxial chromatic aberration (per unit increase in beam energy).

If we are calculating not the chromatic aberration but the relativistic correction, we must replace (9·10) by

$$\left.\begin{aligned} u_b^{(1)R} &= T_1^* u_o + T_2^* v_o + T_3^* u_a + T_4^* v_a, \\ v_b^{(1)R} &= T_5^* u_o + T_6^* v_o + T_7^* u_a + T_8^* v_a,\end{aligned}\right\} \tag{9·12}$$

where $T_1^*, \ldots, T_8^*$ are now the coefficients of the paraxial relativistic correction.

We see from (9·10) that T_1, T_2, T_5 and T_6 characterize the chromatic variation of the magnification. If we suppose, for simplicity, that $f_b = g_b$, there is a uniform expansion and rotation of the image if $T_1 = T_6$ and $T_2 = -T_5$; the fractional expansion is $\epsilon T_1/f_b$, and the angle of rotation is $\epsilon T_5/f_b$. The coefficients T_3, T_4, T_7 and T_8 characterize the defocusing effect. If $f_b = g_b$ and $Q_u = Q_v$, so that $\alpha_b = \beta_b$ also, then $T_4 = T_7$; if $T_3 = -T_8$, a circular aperture of radius r_a will produce a circular image of a point object of radius $\epsilon r_a \surd(T_3^2 + T_4^2)$. If $T_4 = 0$, then $T_7 = 0$ and chromatic variation will split the focus of a point object into two foci distant $-\epsilon T_3/h_b'$ and $-\epsilon T_8/k_b'$ behind the image plane; if $T_3/h_b' = T_8/k_b'$, there is a pure displacement of the focus without splitting.

10. Example: a helical beam in an electric field

It is now proposed that, as a simple example of the application of the theory set out in the present paper, we should derive the imaging properties of an electron beam which moves in a helix in the field of a pair of coaxial cylindrical electrodes. By making the radius and the diametral pitch equal, we shall obtain Gaussian image-formation.

The co-ordinate system is shown in figure 2. If we take the radius and the pitch to be a and $2\pi a$ respectively, then

$$\kappa(z) = \tau(z) = \tfrac{1}{2}a^{-1}. \tag{10·1}$$

Since the field is purely electric, formulae appropriate to the present problem are obtained by setting $\mathsf{E} = 1$ and $\mathsf{M} = 0$. Also we may restrict ourselves to a non-relativistic treatment by replacing all terms enclosed in angular brackets by unity. It will not be necessary to employ the units of § 4, since we shall not investigate the relativistic correction.

The only non-zero field coefficients in the problem are Φ, Φ_x, Φ_{xx} and Φ_{xxx}, and among these there exist the relations

$$\Phi_{xx} = 2\kappa\Phi_x \quad \text{and} \quad \Phi_{xxx} = 8\kappa^2\Phi_x. \tag{10·2}$$

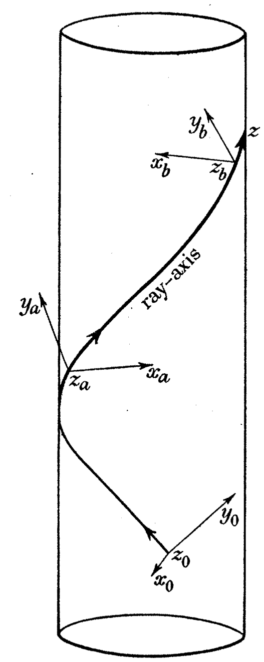

FIGURE 2.

All coefficients are independent of z. We find from (3·7) that the potential in the neighbourhood of the axis is given by

$$\phi(x,y,z) = \Phi + \{x + \kappa x^2 - \tfrac{1}{2}\kappa y^2 + \tfrac{4}{3}\kappa^2 x^3 - 2\kappa^2 x y^2\}\,\Phi_x. \tag{10·3}$$

It may in this way be verified that the coefficients adopted indeed represent the field between cylindrical electrodes for the field of the latter is given, in the present co-ordinate system, by

$$\phi(x,y,z) = \Phi - \tfrac{1}{2}a\Phi_x\{\log\{(a-x)^2 + \tfrac{1}{2}y^2\} - 2\log a\}, \tag{10·4}$$

and (10·3) may be derived from (10·4).

The equation (4·13) now becomes

$$p = \sqrt{(2\Phi)}, \tag{10·5}$$

and the relations (5·4) reduce to the relation

$$\Phi_x = p^2\kappa = a^{-1}\Phi. \tag{10·6}$$

Let us now investigate the paraxial properties. The formulae (6·2) reduce to

$$t = -2p\kappa, \quad q = r = s = 0, \tag{10·7}$$

so that the orthogonality condition (6·9) is obviously satisfied. Since r and $q-s$ both vanish, the paraxial focusing is Gaussian and χ is determined by (6·6). We may therefore write

$$\chi(z) = -\kappa z. \tag{10·8}$$

The principal axes are therefore related to the original axes by

$$\left.\begin{aligned} u &= x\cos\kappa z - y\sin\kappa z,\\ v &= x\sin\kappa z + y\cos\kappa z.\end{aligned}\right\} \tag{10·9}$$

Let us choose as the co-ordinates of the object and aperture planes $z_o = 0$ and $z_a = (\frac{1}{2}\pi)\,\kappa^{-1} = \pi a$. Since, from (6·10),

$$U = V = -p\kappa^2, \tag{10·10}$$

the functions which determine the general paraxial ray are

$$f(z) = g(z) = \cos\kappa z, \quad h(z) = k(z) = \sin\kappa z, \tag{10·11}$$

so that the equations of the general paraxial ray are

$$\left.\begin{aligned} u(z) &= u_o\cos\kappa z + u_a\sin\kappa z,\\ v(z) &= v_o\cos\kappa z + v_a\sin\kappa z.\end{aligned}\right\} \tag{10·12}$$

It is clear that if $z_b = \pi\kappa^{-1} = 2\pi a$, the focusing conditions (6·17) hold; the paraxial magnification is -1.

The constants and functions defined by (6·15) and (6·16) are found to have the values

$$Q_u = Q_v = -p\kappa \tag{10·13}$$

and

$$\left.\begin{aligned} \alpha(z) = \beta(z) &= -p^{-1}\kappa^{-1}\cos\kappa z,\\ \gamma(z) = \theta(z) &= -p^{-1}\kappa^{-1}\sin\kappa z.\end{aligned}\right\} \tag{10·14}$$

The zero-order chromatic aberration may be found from the formulae of § 7. We find from (7·2) that

$$A_1 = -2p^{-1}\kappa, \quad A_2 = 0, \tag{10·15}$$

so that, from (7·5),

$$B_1 = -2p^{-1}\kappa\cos\kappa z, \quad B_2 = -2p^{-1}\kappa\sin\kappa z. \tag{10·16}$$

The integrals (7·8) then give the expressions

$$\left.\begin{aligned} C_{1a}(z) &= \tfrac{1}{2}p^{-1}\{\pi - 2\kappa z - \sin 2\kappa z\},\\ C_{2a}(z) &= \tfrac{1}{2}p^{-1}\{1 + \cos 2\kappa z\},\\ C_{3o}(z) &= -\tfrac{1}{2}p^{-1}\{1 - \cos 2\kappa z\},\\ C_{4o}(z) &= -\tfrac{1}{2}p^{-1}\{2\kappa z - \sin 2\kappa z\},\end{aligned}\right\} \tag{10·17}$$

from which, with the help of (7·11), we obtain the results

$$\left.\begin{aligned} D_1(z) &= p^{-2}\kappa^{-1}\{\tfrac{1}{2}\pi - \kappa z\}\sin\kappa z,\\ D_2(z) &= p^{-2}\kappa^{-1}\kappa z\cos\kappa z.\end{aligned}\right\} \tag{10·18}$$

Hence, or from (7·13),

$$E_1 = 0 \quad \text{and} \quad E_2 = -\pi p^{-2}\kappa^{-1}, \tag{10·19}$$

so that
$$u_b^{(0)\text{I}} = 0, \quad v_b^{(0)\text{I}} = -\pi a \Phi^{-1}, \tag{10·20}$$
which shows that there is no zero-order chromatic shift in the u-direction.

Let us now proceed to evaluate the second-order aberration. The formulae (8·3) give for the non-zero coefficients the simple expressions
$$G_1 = p\kappa, \quad G_7 = \tfrac{1}{3}p\kappa^3, \tag{10·21}$$
from which, by means of (8·2), we find that
$$\left.\begin{aligned} F_1 &= p\kappa\cos\kappa z, & F_5 &= \tfrac{1}{3}p\kappa^3\cos 3\kappa z,\\ F_2 &= p\kappa\sin\kappa z, & F_6 &= p\kappa^3\sin 3\kappa z,\\ F_3 &= 0, & F_7 &= -p\kappa^3\cos 3\kappa z,\\ F_4 &= 0, & F_8 &= -\tfrac{1}{3}p\kappa^3\sin 3\kappa z.\end{aligned}\right\} \tag{10·22}$$
It is proposed that the second-order aberration and the paraxial chromatic aberration be found only for the object point $u_0 = v_0 = 0$, so that only the purely aperture-dependent coefficients of these aberrations will be evaluated.

For this purpose we require the following functions which are found from the formulae (8·6):
$$\left.\begin{aligned} K_{8a}(z) &= \tfrac{1}{48}p\kappa^2\{15\sin 2\kappa z + 6\sin 4\kappa z - \sin 6\kappa z\},\\ K_{9a}(z) &= \tfrac{1}{24}p\kappa^2\{7 + 3\cos 2\kappa z - 3\cos 4\kappa z + \cos 6\kappa z\},\\ K_{10a}(z) &= \tfrac{1}{48}p\kappa^2\{24(\kappa z - \tfrac{1}{2}\pi) + 9\sin 2\kappa z + \sin 6\kappa z\},\\ K_{14a}(z) &= \tfrac{1}{48}p\kappa^2\{-5 - 9\cos 2\kappa z - 3\cos 4\kappa z + \cos 6\kappa z\},\\ K_{15a}(z) &= \tfrac{1}{24}p\kappa^2\{-3\sin 2\kappa z + \sin 6\kappa z\},\\ K_{16a}(z) &= \tfrac{1}{48}p\kappa^2\{11 + 9\cos 2\kappa z - 3\cos 4\kappa z - \cos 6\kappa z\},\\ K_{17o}(z) &= \tfrac{1}{144}p\kappa^2\{17 - 9\cos 2\kappa z - 9\cos 4\kappa z + \cos 6\kappa z\},\\ K_{18o}(z) &= \tfrac{1}{48}p\kappa^2\{9\sin 2\kappa z - 6\sin 4\kappa z + \sin 6\kappa z\},\\ K_{19o}(z) &= \tfrac{1}{48}p\kappa^2\{13 - 15\cos 2\kappa z + 3\cos 4\kappa z - \cos 6\kappa z\},\\ K_{20o}(z) &= \tfrac{1}{144}p\kappa^2\{24\kappa z - 9\sin 2\kappa z - \sin 6\kappa z\}.\end{aligned}\right\} \tag{10·23}$$
We now find from (8·10) that
$$\left.\begin{aligned} L_8(z) &= \tfrac{1}{96}\kappa\{-10\cos\kappa z + 9\cos 3\kappa z + \cos 5\kappa z\},\\ L_9(z) &= \tfrac{1}{48}\kappa\{2\sin\kappa z + 3\sin 3\kappa z + \sin 5\kappa z\},\\ L_{10}(z) &= \tfrac{1}{96}\kappa\{48(\kappa z - \tfrac{1}{2}\pi)\sin\kappa z - 2\cos\kappa z + 3\cos 3\kappa z - \cos 5\kappa z\},\\ L_{18}(z) &= \tfrac{1}{96}\kappa\{-10\sin\kappa z - 9\sin 3\kappa z + \sin 5\kappa z\},\\ L_{19}(z) &= \tfrac{1}{48}\kappa\{-14\cos\kappa z + 15\cos 3\kappa z - \cos 5\kappa z\},\\ L_{20}(z) &= \tfrac{1}{96}\kappa\{-48\kappa z\cos\kappa z + 22\sin\kappa z + 21\sin 3\kappa z - \sin 5\kappa z\},\end{aligned}\right\} \tag{10·24}$$
and hence, or from (8·12), that
$$M_8 = M_9 = M_{10} = M_{18} = M_{19} = 0, \quad M_{20} = \tfrac{1}{2}\pi\kappa. \tag{10·25}$$
The second-order aberration for the central object point is therefore given by
$$u_b^{(2)} = 0, \quad v_b^{(2)} = \tfrac{1}{4}\pi a^{-1} v_a^2, \tag{10·26}$$

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

so that the second-order aperture aberration has the effect of lengthening the image of a point object into a line parallel to the v-axis, which extends to the 'positive' side of the paraxial image point and whose length is proportional to the square of the linear dimensions of the aperture.

We need also the values of the following coefficients which appear in (8·14):

$$N_{18} = \tfrac{1}{3}p\kappa^2, \quad N_{19} = 0, \quad N_{20} = -\tfrac{1}{3}p\kappa^3. \tag{10·27}$$

The paraxial chromatic aberration may be evaluated by the formulae set out in § 9. The functions given by (9·3) reduce to

$$\left.\begin{aligned} R_1 &= \tfrac{1}{2}p^{-1}, \\ R_2 &= 0, \\ R_3 &= p^{-1}\kappa^2, \\ R_4 &= \tfrac{1}{2}p^{-1}\kappa^2, \\ R_5 &= 0, \end{aligned}\right\} \tag{10·28}$$

so that we obtain, with the help of (9·2),

$$\left.\begin{aligned} P_1 &= \tfrac{1}{2}p^{-1}, \\ P_2 &= 0, \\ P_3 &= \tfrac{1}{2}p^{-1}\kappa^2(2+\cos 2\kappa z), \\ P_4 &= p^{-1}\kappa^2 \sin 2\kappa z, \\ P_5 &= \tfrac{1}{2}p^{-1}\kappa^2(2-\cos 2\kappa z). \end{aligned}\right\} \tag{10·29}$$

Since we wish to obtain the paraxial chromatic aberration only for the object point $u_o = v_o = 0$, we need find only S_8, S_9 and S_{10}. Using the formulae (9·8) and with the help of (10·10) and (10·24), we find that

$$S_8 = \tfrac{5}{6}\pi p^{-1}\kappa, \quad S_9 = 0, \quad S_{10} = \tfrac{7}{6}\pi p^{-1}\kappa. \tag{10·30}$$

The formulae (9·11), with (10·19) and (10·27), then give the result

$$\left.\begin{aligned} T_3 &= \pi p^{-2}, \quad & T_7 &= 0, \\ T_4 &= 0, & T_8 &= 3\pi p^{-2}, \end{aligned}\right\} \tag{10·31}$$

from which we see that the aperture-dependent part of the paraxial chromatic aberration is given by

$$u_b^{(1)\mathrm{I}} = \tfrac{1}{2}\pi\Phi^{-1}u_a, \quad v_b^{(1)\mathrm{I}} = \tfrac{3}{2}\pi\Phi^{-1}v_a. \tag{10·32}$$

Since $S_9 = 0$ but $T_3 \neq T_8$, the paraxial chromatic aperture aberration may be interpreted as a splitting of the focus of an object point into a pair of foci. If the increase in beam energy is ϵ, the u- and v-foci are distant $\epsilon\pi a\Phi^{-1}$ and $3\epsilon\pi a\Phi^{-1}$, respectively, behind the image plane.

Let us now gather together the results obtained for the present problem and express them in terms of the original co-ordinate system. We find from (10·12), (10·20), (10·26) and (10·32) that the co-ordinates of a ray in the image plane are related to their co-ordinates in the object and aperture plane by

$$\left.\begin{aligned} u_b &= -u_o + \tfrac{1}{2}\epsilon\pi\Phi^{-1}u_a, \\ v_b &= -v_o + \tfrac{1}{4}\pi a^{-1}v_a^2 - \epsilon\pi a\Phi^{-1} + \tfrac{3}{2}\epsilon\pi\Phi^{-1}v_a, \end{aligned}\right\} \tag{10·33}$$

to the approximations adopted in this paper. If we now transform according to (10·9), we obtain

$$\left.\begin{aligned} x_b &= x_o + \tfrac{1}{2}\epsilon\pi\Phi^{-1}y_a, \\ y_b &= y_o - \tfrac{1}{4}\pi a^{-1}x_a^2 + \epsilon\pi a\Phi^{-1} - \tfrac{3}{2}\epsilon\pi\Phi^{-1}x_a. \end{aligned}\right\} \qquad (10\cdot34)$$

It is interesting to investigate whether we can so choose the aperture position that T_3 and T_8 become equal. It is easily found from the equations (A 4) of the appendix that $\overline{T}_3 = \overline{T}_8$ if $z_{\bar{a}}$ is a root of the equation $2\tan z_{\bar{a}} = z_{\bar{a}}$, a particular solution of which is $z_{\bar{a}} = 0$. Upon a chromatic variation the focus, which is not split, is then displaced a distance $\epsilon\pi a\Phi^{-1}$ behind the image plane. On combining our calculations of the zero-order and paraxial chromatic aberrations, we find that the focus is displaced from $(0, 0, z_b)$ to $(0, \epsilon\pi a\Phi^{-1}, z_b + \epsilon\pi a\Phi^{-1})$, which clearly represents a displacement of the image point by a distance $\epsilon\pi\sqrt{2}\,a\Phi^{-1}$ in a direction parallel to the axis of the cylindrical electrodes.

The above result has been obtained by Gabor (1951) in his elegant investigation of the chromatic aberration of helical systems. However, on studying the arguments by which Gabor's results are obtained, one must regard the agreement as fortuitous. According to Gabor's calculations, the result depends neither on the choice of aperture position nor on the values of Φ_{xxx} and Φ_{xxy}, whereas it is clear from the present treatment that these factors both influence the paraxial chromatic aberration.

If we calculate the paraxial chromatic aberration of the circular beam of radius r, with which Gabor commences his study, we find that

$$\left.\begin{aligned} T_1 &= \tfrac{1}{2}\pi(13 - r^3\Phi_{xxx}/\Phi), & T_2 &= 0, \\ T_3 &= 0, & T_4 &= \tfrac{1}{2}\pi(-11 + r^3\Phi_{xxx}/\Phi), \end{aligned}\right\}$$

so that there is no splitting of the focus only if $\Phi_{xxx} = 12\Phi/r^3$; the longitudinal shift is then $-\tfrac{1}{2}\epsilon\pi r\Phi^{-1}$, which disagrees with Gabor's value of $-3\epsilon\pi r\Phi^{-1}$. This example is interesting in that the zero-order chromatic aberration and the second-order aperture aberration both vanish so that the aperture-dependent part of the paraxial chromatic aberration is independent of the aperture position; this is exceptional.

The error in Gabor's calculation is to estimate the chromatic shift of focus by considering the behaviour of an electron following the central ray only of a beam, for the term 'focus' refers to a *pencil* of rays, not to a single ray. It should also be remarked that the use of the term 'chromatic shift of focus' must in each case be justified—as we have already seen—since, upon chromatic variation, the focus may be lost or may split into a pair of foci.

Appendix. Change of aperture position

In the foregoing calculations, it has been assumed that the aperture is specified from the outset. However, one may decide to leave the aperture position arbitrary until its influence upon the aberrations can be assessed. We shall therefore investigate in this appendix how one may calculate the change in the imaging properties of a system due to a change in its aperture position.

The co-ordinates of a ray in the image plane are related to its co-ordinates in the object and aperture planes by (1·3), i.e.

$$\left.\begin{aligned} u_b &= u_o f_b + \epsilon u_b^{(0)\mathrm{I}} + u_b^{(2)} + \epsilon u_b^{(1)\mathrm{I}}, \\ v_b &= v_o g_b + \epsilon v_b^{(0)\mathrm{I}} + v_b^{(2)} + \epsilon v_b^{(1)\mathrm{I}}, \end{aligned}\right\} \qquad (\mathrm{A\,1})$$

where $u_b^{(0)\mathrm{I}}$, $v_b^{(0)\mathrm{I}}$, $u_b^{(2)}$, $v_b^{(2)}$ and $u_b^{(1)\mathrm{I}}$, $v_b^{(1)\mathrm{I}}$ are given by (7·12), (8·11) and (9·10). Let us now suppose that, upon changing the aperture from $z = z_a$ to $z = z_{\bar{a}}$, all the coefficients f_b, E_1, etc., appearing in these expansions are changed to $\bar{f}_b$, $\bar{E}_1$, etc. We then obtain a pair of equations similar to (A 1) which relate u_b, v_b to u_o, v_o, $u_{\bar{a}}$, $v_{\bar{a}}$.

If we take into account the second-order aberration and the zero-order chromatic aberration but ignore the paraxial chromatic aberration, we have the following expressions for the ray co-ordinates along the axis:

$$\left.\begin{aligned} u(z) &= u_o f(z) + u_a h(z) + \epsilon u^{(0)\mathrm{I}}(z) + u^{(2)}(z), \\ v(z) &= v_o g(z) + v_a k(z) + \epsilon v^{(0)\mathrm{I}}(z) + v^{(2)}(z), \end{aligned}\right\} \tag{A 2}$$

where $u^{(0)\mathrm{I}}(z)$, $v^{(0)\mathrm{I}}(z)$ and $u^{(2)}(z)$, $v^{(2)}(z)$ are given by (7·10) and (8·9). The equations (A 2) may be used to relate $u_{\bar{a}}$, $v_{\bar{a}}$ to u_o, v_o, u_a, v_a, and so to compare our two sets of equations for u_b, v_b.

In this way it is possible to express the barred coefficients in terms of the unbarred coefficients. Apart from the trivial results $f_b = \bar{f}_b$, $\bar{g}_b = g_b$, $\bar{E}_1 = E_1$ and $\bar{E}_2 = E_2$, we find that

$$\left.\begin{aligned} \bar{M}_1 &= M_1 - M_3 f_{\bar{a}}/h_{\bar{a}} + M_8 f_{\bar{a}}^2/h_{\bar{a}}^2, \\ \bar{M}_2 &= M_2 - M_4 g_{\bar{a}}/k_{\bar{a}} - M_6 f_{\bar{a}}/h_{\bar{a}} + M_9 f_{\bar{a}} g_{\bar{a}}/h_{\bar{a}} k_{\bar{a}}, \\ \bar{M}_3 &= M_3/h_{\bar{a}} - 2M_8 f_{\bar{a}}/h_{\bar{a}}^2, \\ \bar{M}_4 &= M_4/k_{\bar{a}} - M_9 f_{\bar{a}}/h_{\bar{a}} k_{\bar{a}}, \\ \bar{M}_5 &= M_5 - M_7 g_{\bar{a}}/k_{\bar{a}} + M_{10} g_{\bar{a}}^2/k_{\bar{a}}^2, \\ \bar{M}_6 &= M_6/h_{\bar{a}} - M_9 g_{\bar{a}}/h_{\bar{a}} k_{\bar{a}}, \\ \bar{M}_7 &= M_7/k_{\bar{a}} - 2M_{10} g_{\bar{a}}/k_{\bar{a}}^2, \\ \bar{M}_8 &= M_8/h_{\bar{a}}^2, \\ \bar{M}_9 &= M_9/h_{\bar{a}} k_{\bar{a}}, \\ \bar{M}_{10} &= M_{10}/k_{\bar{a}}^2 \end{aligned}\right\} \tag{A 3}$$

and

$$\left.\begin{aligned} \bar{T}_1 &= T_1 - T_3 f_{\bar{a}}/h_{\bar{a}} - M_3 D_{1\bar{a}}/h_{\bar{a}} - M_4 D_{2\bar{a}}/k_{\bar{a}} + 2M_8 D_{1\bar{a}} f_{\bar{a}}/h_{\bar{a}}^2 + M_9 D_{2\bar{a}} f_{\bar{a}}/h_{\bar{a}} k_{\bar{a}}, \\ \bar{T}_2 &= T_2 - T_4 g_{\bar{a}}/k_{\bar{a}} - M_6 D_{1\bar{a}}/h_{\bar{a}} - M_7 D_{2\bar{a}}/k_{\bar{a}} + M_9 D_{1\bar{a}} g_{\bar{a}}/h_{\bar{a}} k_{\bar{a}} + 2M_{10} D_{2\bar{a}} g_{\bar{a}}/k_{\bar{a}}^2, \\ \bar{T}_3 &= T_3/h_{\bar{a}} - 2M_8 D_{1\bar{a}}/h_{\bar{a}}^2 - M_9 D_{2\bar{a}}/h_{\bar{a}} k_{\bar{a}}, \\ \bar{T}_4 &= T_4/k_{\bar{a}} - M_9 D_{1\bar{a}}/h_{\bar{a}} k_{\bar{a}} - 2M_{10} D_{2\bar{a}}/k_{\bar{a}}^2. \end{aligned}\right\} \tag{A 4}$$

The corresponding formulae for $\bar{M}_{11}, \ldots, \bar{M}_{20}$ and $\bar{T}_5, \ldots, \bar{T}_8$ are obtained by replacing $M_1, \ldots, M_{10}$ and $T_1, \ldots, T_4$ by $M_{11}, \ldots, M_{20}$ and $T_5, \ldots, T_8$, respectively. The relations (A 4) demonstrate most clearly the dependence of the coefficients of paraxial chromatic aberration upon the coefficients of zero-order chromatic aberration and second-order aberration.

We may find, in the same way, the change in the coefficients appearing in (A 2) due to a change in aperture position. It is found that

$$\left.\begin{aligned} \bar{f}(z) &= f(z) - f_{\bar{a}} h(z)/h_{\bar{a}}, \quad & \bar{h}(z) &= h(z)/h_{\bar{a}}, \\ \bar{g}(z) &= g(z) - g_{\bar{a}} k(z)/k_{\bar{a}}, \quad & \bar{k}(z) &= k(z)/k_{\bar{a}}, \end{aligned}\right\} \tag{A 5}$$

and

$$\bar{D}_1(z) = D_1(z) - D_{1\bar{a}} h(z)/h_{\bar{a}}, \quad \bar{D}_2(z) = D_2(z) - D_{2\bar{a}} k(z)/k_{\bar{a}}, \tag{A 6}$$

and that the formulae for $\bar{L}_1(z)$, etc., may be obtained from (A 3) by replacing M_r by $L_r(z) - L_{r\bar{a}} h(z)/h_{\bar{a}}$ for $r = 1, \ldots, 10$ and by $L_r(z) - L_{r\bar{a}}(z)/k_{\bar{a}}$ for $r = 11, \ldots, 20$.

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